PHYS 725 Final Examination
11 December 2001

Solutions

1. The equation of an ellipsoidal surface in 3 dimensions is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\,.$$

Calculate the volume enclosed by this surface. (Show your work!)
Solution:

Method I:

$$V=\int_{-c}^{c}dz\int_{-b}^{b}dy\int_{-a}^{a}dx\,\theta\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right)$$

hence x runs from

$$-a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\quad\text{to}\quad a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\,.$$

Doing the $x-$integral we then have

$$V=2a\int_{-c}^{c}dz\int_{-b}^{b}dy\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\,.$$

But y now runs from

$$-b\sqrt{1-\frac{z^2}{c^2}}\quad to\quad b\sqrt{1-\frac{z^2}{c^2}}$$

so we have

$$\begin{aligned} V &= 2a\int_{-c}^{c}dz\left(1-\frac{z^2}{c^2}\right)\int_{-b}^{b}dy\sqrt{1-\frac{y^2}{b^2}} \\ &= 8abc\int_{0}^{1}d\zeta\left(1-\zeta^2\right)\int_{0}^{1}d\eta\sqrt{1-\eta^2}=\frac{4\pi}{3}abc\,. \end{aligned}$$

Method II:

$$\begin{aligned} x &= a\rho \sin\theta \cos\varphi \\ y &= b\rho \sin\theta \sin\varphi \\ z &= c\rho \cos\theta\,; \end{aligned}$$

these manifestly lie on the surface when $\rho = 1$. The Jacobian of the transformation is easily see to be

$$dx\,dy\,dz = abc\rho^2 d\rho\,\sin\theta\,d\theta\,d\varphi$$

so the volume becomes

$$V = abc\int_0^1 \rho^2 d\rho\,\int_0^{2\pi} d\varphi\,\int_0^{\pi} \sin\theta\,d\theta = \frac{4\pi}{3}abc\,.$$

2. Laguerre polynomials $L_n(x)$ are defined on the interval $0 \le x < +\infty$, and satisfy the orthogonality relation

$$\int_0^\infty dx\,L_m\,(x)\,L_n\,(x)e^{-x} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$$

Apply the Gram-Schmidt orthogonalization method to the monomials $x^0, x^1, x^2, \ldots$ in order to derive the Laguerre polynomials $L_1(x)$, $L_2(x)$, with $L_0(x) = 1$.
Solution:

$$L_0(x) = 1$$

This is already normalized, since

$$\int_0^\infty 1 \cdot e^{-x} dx = 1$$

so

$$L_1\,(x) = a\left(x - L_0(x) \cdot \int_0^\infty x' \cdot L_0(x') \cdot e^{-x'} dx'\right) = a\,(x-1)\ .$$

We evaluate the unknown normalization constant from the integral

$$\int_0^\infty (L_1(x))^2 \, e^{-x} dx = a^2 \int_0^\infty (x-1)^2 \, e^{-x} dx = a^2 \, (2! - 2 + 1) = 1$$

or $a = \pm 1$. To get L_2 we note that

$$\begin{aligned} L_2\,(x) = a \left[x^2 - (x-1) \int_0^\infty (x'-1)\, x'^2 \, e^{-x'} dx' - 1 \cdot \int_0^\infty x'^2 \, e^{-x'} dx' \right] \\ = a \left[x^2 - 4\,(x-1) - 2 \right] = a \left(x^2 - 4x + 2 \right) \end{aligned}$$

Normalizing,

$$\begin{aligned} \int_0^\infty (L_2(x))^2 \, e^{-x} dx &= a^2 \int_0^\infty \left(x^2 - 4x + 2 \right)^2 \, e^{-x} dx \\ &\equiv a^2 \int_0^\infty \left(x^2 - 4x + 2 \right) x^2 \, e^{-x} dx \quad \text{(Why?)} \\ &= a^2 \, (4! - 4 \cdot 3! + 2 \cdot 2!) = 4a^2 = 1 \end{aligned}$$

or $a = \pm 1/2$.

3. Evaluate the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - 0.25}$$

in closed form, using any method that seems promising.
Solution:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^2 - 0.25} = \lim_{N \to \infty} \sum_{n=1}^{N} \left(\frac{1}{n - 0.5} - \frac{1}{n + 0.5} \right) \\ = \lim_{N \to \infty} \left(\frac{1}{1 - 0.5} - \frac{1}{N + 0.5} \right) = 2 \end{aligned}$$

Alternatively, we can use contour integration or the formula

$$-\pi \cot(\pi \lambda) = -\frac{1}{\lambda} + 2\lambda \sum_{n=1}^{\infty} \frac{1}{n^2 - \lambda^2}$$

Let $\lambda = 0.5$; then since $\cot(\pi/2) = 0$ we have

$$0 = -2 + \sum_{n=1}^{\infty} \frac{1}{n^2 - 0.25}$$

QED.

4. The Ruritanian zither is a single-stringed instrument, whose string has a radius that varies with linear position as

$$R(x) = R_0 \sqrt{1 + \tfrac{1}{4}\sin(\pi x/L)}\,,$$

where L is the length of the string.

(a) (10 points) Derive the (partial differential) equation of motion of the string. Assume the string is made of material with uniform volumetric mass-density ρ.

Solution:

As discussed in class, and as is done on p. 352-354 of the notes, we break the string into lumps of mass $\Delta m = \mu(x)\,\Delta x = \rho\pi R^2 \Delta x$ and apply Newton's second law to the displacement of the n'th mass:

$$\Delta m \frac{d^2\psi_n}{dt^2} = -T\frac{\psi_n - \psi_{n-1}}{\Delta x} - T\frac{\psi_n - \psi_{n+1}}{\Delta x}\,.$$

Going to the continuum limit, $\Delta x \to 0$, we find

$$\frac{\mu(x)}{T}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x^2}\,,$$

where $\mu(x) = \rho\pi R^2(x)$.

(b) (10 points) If the string is clamped at both ends, estimate the frequency of the lowest vibrational mode of the string, in terms of the tension T, length L, radius R_0 and density ρ.

Solution:

Applying separation of variables (and skipping a step!) we have

$$\psi(x,t) = \phi(x)\,e^{i\omega t}$$

where

$$\omega^2 \frac{\mu(x)}{T} \phi(x) = -\frac{d^2\phi}{dx^2} \, .$$

Therefore we may multiply both sides by $\phi(x)$ and integrate from 0 to L and obtain

$$\omega^2 = \frac{\int_0^L dx \left[\frac{d\phi}{dx}\right]^2}{\int_0^L dx \left[\phi(x)\right]^2 \frac{\mu(x)}{T}} = F(\{\phi\}) \, .$$

However, as discussed in class, the lowest frequency is bounded above by $F(\{\chi\})$ where χ is *any* function that vanishes at both endpoints. So let us take $\chi(x) = \sin(\pi x/L)$ and evaluate. We get

$$\begin{aligned} \omega_0^2 &\le F(\{\chi\}) = \frac{\pi^2}{L^2} \left(\frac{T}{\rho\pi R_0^2}\right) \frac{\int_0^\pi d\theta \, \cos^2\theta}{\int_0^\pi d\theta \left[1 + \frac{1}{4}\sin\theta\right] \sin^2\theta} \\ &= \frac{\pi^2}{L^2} \left(\frac{T}{\rho\pi R_0^2}\right) \frac{1}{1 + \frac{2}{3\pi}} \end{aligned}$$

5. The rate of heat flow in an isotropic solid can be defined in terms of a flux vector (thermal energy per unit area per unit time across a surface normal to the vector)

   $$\vec{\jmath}_Q = -\kappa \nabla T \, ,$$

   where κ is the thermal conductivity. (This relation was deduced by Isaac Newton!)

   The heat energy is conserved (First Law of Thermodynamics)

   $$\frac{\partial U_Q}{\partial t} + \nabla \cdot \vec{\jmath}_Q = 0$$

   and we may assume the heat energy density is linear in the temperature,

   $$U_Q = c_V T \, ,$$

   where the constant of proportionality c_V is the specific heat (per unit volume).

(a) (15 points) Use this information to derive an equation for the temperature distribution in the body, as a function of position and time.

Solution:

$$\nabla \cdot \vec{j}_Q = -\kappa \nabla^2 T$$

and thus

$$\frac{\partial U_Q}{\partial t} = \kappa \nabla^2 T$$

or

$$\frac{\partial T}{\partial t} = \frac{\kappa}{c_V} \nabla^2 T$$

giving

$$D = \kappa / c_V \, .$$

(b) (5 points) A long thin rod, of cross-section A and length L is initially at $T = 300\,^\circ K$ and one end is placed in a furnace at $T = 500\,^\circ K$. Two (2) seconds later, the other end of the rod has reached a temperature of $T = 450\,^\circ K$. What would the time be for a similar rod of length $2L$?

Solution:

As we discussed at great length in class, and in p. 355ff of the online lecture notes, in diffusive processes distance scales as $t^{0.5}$. Or we could see this from dimensional analysis of the diffusion equation itself:

$$\left[\frac{\partial T}{\partial t}\right] = \frac{[T]}{time}$$

$$\left[D \nabla^2 T\right] = [D] \, \frac{[T]}{length^2}$$

or

$$t \propto D^{-1} \ell^2 \, .$$

Thus the time for a bar twice as long to reach the same temperature must be 4× the previous time, or in this case, 8 seconds.

(Possibly) Useful Formulae

$$\int_{-\infty}^{\infty} dx e^{ikx} e^{-x^2} = \sqrt{\pi}\, e^{-k^2/4}$$

(Note: you can get $\int_{-\infty}^{\infty} dx\, x^{2n} e^{-x^2}$ by comparing the series expansions of $e^{-k^2/4}$ and e^{ikx} on both sides.

If $f(z)$ is analytic within Γ, then $\oint_\Gamma dz f(z) = 0\,.$

If $z^n = 1$ then $z = e^{2k\pi i/n},\; k = 0, 1, \ldots, n-1\,.$

$$\begin{aligned} e^z &= 1 + \tfrac{z}{1!} + \tfrac{z^2}{2!} + \ldots \\ \cos z &= 1 - \tfrac{z^2}{2!} + \tfrac{z^4}{4!} - + \ldots \\ \sin z &= \tfrac{z}{1!} - \tfrac{z^3}{3!} + - \ldots \end{aligned}$$

$$\begin{aligned} de^z &= e^z dz & d\cos z &= -\sin z\, dz \\ d\sin z &= \cos z\, dz & d\tan z &= \sec^2 z\, dz \\ d\sinh z &= \cosh z\, dz & d\cosh z &= \sinh z\, dz \\ d\log z &= \tfrac{dz}{z} & d\tan^{-1} z &= \tfrac{dz}{z^2+1} \end{aligned}$$

$$\Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t}, \qquad \Gamma(z+1) = z\Gamma(z)$$

$$B(v,w) = \int_0^1 dt\, t^{v-1} (1-t)^{w-1} \equiv \frac{\Gamma(v)\, \Gamma(w)}{\Gamma(v+w)}$$

Bessel's equation: $x^2\psi'' + x\psi' + \left(x^2 - m^2\right)\psi = 0$

if $\psi(x) = x^\alpha J_{\pm m}(\beta x^\gamma)$,

then $x^2\psi'' + x(1-2\alpha)\psi' + \left(\beta^2\gamma^2 x^{2\gamma} + \alpha^2 - m^2\gamma^2\right)\psi = 0$