PHYS 725 Final Examination
11 December 2001

Note: Time limit 3 hours. Answer all 5 questions. All questions are weighted equally, although they may be of varying difficulty.

1. The equation of an ellipse in 2-dimensional radial (cylindrical) coordinates r, θ is

$$\frac{\ell}{r(\theta)} = 1 - \varepsilon \cos\theta \, ,$$

where ℓ and ε are constants ($\varepsilon < 1$); what is its area?

2. Chebyshev polynomials $T_n(x)$ are defined on the interval $-1 \leq x \leq 1$, and satisfy the orthogonality relation

$$\int_{-1}^{1} dx \, \frac{T_m(x)\, T_n(x)}{\sqrt{1-x^2}} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$$

Apply the Gram-Schmidt orthogonalization method to the monomials $x^0, x^1, x^2, \ldots$ in order to derive the Chebyshev polynomials $T_2(x)$, $T_3(x)$, with $T_0(x) = 1$.

3. Evaluate the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^4 + 1}$$

in closed form.

4. We are interested in sound propagation in an ideal gas. Ignoring viscosity as well as terms of second order in velocity, the equation of motion of a fluid is (Euler's equation)

$$\rho \frac{\partial \vec{v}}{\partial t} = -\nabla p$$

where ρ is the (local) mass density of the fluid, $\vec{v}$ is the local velocity of a small volume of the fluid, and p is the local pressure. (This is clearly just Newton's 2nd Law of Motion, $\vec{F} = m\vec{a}$.) The equation of mass conservation is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \, .$$

An ideal gas also satisfies the equation of state (relating pressure, density and temperature)

$$p = \frac{\rho}{\mu} RT \, .$$

Finally, conservation of energy demands that for adaibatic compression and expansion (no heat flows in or out of a volume in one oscillation)

$$-\mu \, p \frac{d\rho}{\rho^2} + \frac{f}{2} R dT = dQ = 0 \, ,$$

where μ is the mass of a gram-mol (for He it is 4 gm, for O_2 it is 32 gm, *etc.*), R is the gas constant 8.3 Joules/gm-mol/$^\circ K$, and f is the number of degrees of freedom (for a monoatomic gas $f = 3$, for a diatomic gas, $f = 5$, *etc.*).

Imagine the gas has equilibrium pressure $\bar{p}$, density $\bar{\rho}$ and no overall velocity. Then we want to consider "small" fluctuations about the equilibrium values (note we can treat the velocity $\vec{v}$ associated with such fluctuations as small since there is no overall motion), *i.e.*

$$\rho = \bar{\rho} + \delta\rho, \quad p = \bar{p} + \delta p, \quad T = \bar{T} + \delta T$$

Use the above information (equation of motion, conservation of mass, energy conservation, equation of state) to derive the wave equation of sound (keep in mind that the equilibrium values $\bar{p}$ *etc.*are constants!) and identify the speed of sound in terms of R, $\bar{T}$, μ, *etc.*

How does the speed of sound depend on gas pressure? Is this reasonable? In that case, how do you explain the fact that sound travels faster at sea level than at high altitude?

5. The transport of heat in a solid is described by the equation

$$D\nabla^2 T(\vec{x}, t) = \frac{\partial}{\partial t} T(\vec{x}, t)$$

where $T(\vec{x}, t)$ is the spatial distribution of temperature at time t.

Now consider the problem faced by a chef cooking a large beef roast for a banquet. She places the roast, which is initially at room temperature, in an oven at 180° C and would like to know the cooking time based on the mass of the roast. She knows that for a 2 kg roast the time will be 2 hours. How long should a 54 kg roast of similar proportions (that is, simply scaled up in linear dimension) take? (**Hint:** express the equation in dimensionless variables and note that the centers of the two roasts must be at the same temperature when they are done.)

(Possibly) Useful Formulae

$$\int_{-\infty}^{\infty} dx e^{ikx} e^{-x^2} = \sqrt{\pi}\, e^{-k^2/4}$$

(Note: you can get $\int_{-\infty}^{\infty} dx\, x^{2n} e^{-x^2}$ by comparing the series expansions of $e^{-k^2/4}$ and e^{ikx} on both sides.

$$\text{If } f(z) \text{ is analytic within } \Gamma, \text{ then } \oint_{\Gamma} dz f(z) = 0\,.$$

$$\text{If } z^n = 1 \text{ then } z = e^{2k\pi i/n},\ k = 0, 1, \ldots, n-1\,.$$

$$\begin{aligned} e^z &= 1 + \tfrac{z}{1!} + \tfrac{z^2}{2!} + \ldots \\ \cos z &= 1 - \tfrac{z^2}{2!} + \tfrac{z^4}{4!} - + \ldots \\ \sin z &= \tfrac{z}{1!} - \tfrac{z^3}{3!} + - \ldots \end{aligned}$$

$$\begin{aligned} de^z &= e^z dz & d\cos z &= -\sin z\, dz \\ d\sin z &= \cos z\, dz & d\tan z &= \sec^2 z\, dz \\ d\sinh z &= \cosh z\, dz & d\cosh z &= \sinh z\, dz \\ d\log z &= \tfrac{dz}{z} & d\tan^{-1} z &= \tfrac{dz}{z^2+1} \end{aligned}$$

$$\Gamma(z) = \int_0^{\infty} dt\, t^{z-1} e^{-t}, \qquad \Gamma(z+1) = z\Gamma(z)$$

$$B(v, w) = \int_0^1 dt\, t^{v-1} (1-t)^{w-1} \equiv \frac{\Gamma(v)\, \Gamma(w)}{\Gamma(v+w)}$$

$$\text{Bessel's equation: } x^2\psi'' + x\psi' + \left(x^2 - m^2\right)\psi = 0$$

$$\text{if } \psi(x) = x^{\alpha} J_{\pm m}(\beta x^{\gamma}),$$

$$\text{then } x^2\psi'' + x(1-2\alpha)\psi' + \left(\beta^2\gamma^2 x^{2\gamma} + \alpha^2 - m^2\gamma^2\right)\psi = 0$$