Riley, *et. al* p. 667

18.1 We are asked to find an analytic function of $z = x + iy$ whose imaginary part is

$$v(x,y) = \left[y \cos y + x \sin y\right] e^x .$$

We use the Cauchy-Riemann equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \left[\cos y - y \sin y + x \cos y\right] e^x$$

to get

$$u = \left(\cos y - y \sin y\right) e^x + \cos y \left(x e^x - e^x\right) + g(y) .$$

From the other Cauchy-Riemann equation,

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

we find $g'(y) = 0$, *i.e.* $g =$ constant. Thus, putting together u and v we get

$$f(z) = u + iv \equiv z\, e^z .$$

18.2 The answer given in the book on p. 671 is correct but useless, since it is not a function of $z = x + iy$. The answer can be obtained by the same method as above. It satisfies the Cauchy-Riemann equations, so it is an analytic function and therefore expressible as a function of $z = x + iy$. How can we find this function? Calculate the derivative of

$$f(x,y) = \frac{\sin 2x - i \sinh 2y}{\cosh 2y - \cos 2x}$$

by looking at the leading terms of

$$f(x+\delta x, y+\delta y) \approx f(x,y) + \frac{df}{dz}(\delta x + i\delta y) = f(x,y) + \frac{df}{dz}\delta z$$

to find

$$\frac{df}{dz} = 2\, \frac{-1 + \cos 2x \cosh 2y + i \sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2} ;$$

evaluate at $y = 0$: $f'(x) = \dfrac{2}{\cos 2x - 1}$, or

$$f(z) = \int^z dw \frac{2}{\cos 2w - 1} + \text{constant} .$$

Performing the integral we obtain $f(z) = \dfrac{2i}{e^{2iz} - 1} + c$, which is easily seen to equal the book answer, if we set $c = i$. The singularities (poles) occur when $z = 2n\pi$.

18.3 The radii of convergence can be found by the ratio test:

$$R = \lim_{n \to \infty} \frac{a_n}{a_{n+1}}$$

a) $$\frac{\log(n+1)}{\log n} \to \frac{\log\left[n\left(1+\frac{1}{n}\right)\right]}{\log n} = 1 + \frac{1}{n \log n} \to 1$$

b) $$\frac{(n+1)^{n+1}}{(n+1)!}\frac{n!}{n^n} = \exp\left[(n+1)\log(n+1) - n\log n - \log(n+1)\right]$$

$$= \exp\left[n\log\left(1+\frac{1}{n}\right)\right] \to \exp\left[1-\frac{1}{2n}+\ldots\right] \to e$$

c) $$\frac{n^{\log n}}{(n+1)^{\log(n+1)}} \equiv \exp\left[\log^2 n - \left(\log n + \frac{1}{n} + \ldots\right)^2\right] = \exp\left[-\frac{2\log n}{n} - \ldots\right] \to e^0 = 1$$

d) $$\left(\frac{n+p}{n}\right)^{n^2}\left(\frac{n+1}{n+1+p}\right)^{(n+1)^2} \equiv \exp\left[n^2\log\left(1+\frac{p}{n}\right) - (n+1)^2\log\left(1+\frac{p}{n+1}\right)\right]$$

$$= \exp\left[np - \frac{1}{2} + \frac{1}{3n} - (n+1)p + \frac{1}{2} - \frac{1}{3(n+1)}\ldots\right] \to e^{-p}$$

18.4 If $r > \frac{2p|z|}{\pi}$ we can use Jordan's Lemma to say $\left|\sin\left(\frac{pz}{r}\right)\right| < \frac{p|z|}{r}$ hence the terms of the series are then decreasing and alternating in sign and therefore converge by Weierstrass' criterion. This is true for any p and z.

We now calculate the derivatives at the origin:

$$\frac{d}{dz}f(0) = p\sum_1^\infty (-1)^{r+1}\frac{1}{r} = p\log 2$$

$$\frac{d^2}{dz^2}f(0) = 0$$

$$f^{(3)}(0) = p^3\left(1-\frac{1}{8}+\ldots\right)$$

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It is easy to see that the derivatives are the corresponding powers of p with coefficients that are either 0 (even derivatives) or that approach 1. Thus the Taylor series for $f(z)$ has the form

$$f(z) = \sum_{n=0}^{\infty} \frac{c_{2n+1}}{(2n+1)!} (pz)^{2n+1};$$

since the coefficients c_{2n+1} are smaller than 1 and positive the series has an *infinite* radius of convergence.

18.5 Zeros, poles, branch cuts and essential singularities of

a) $\tan z$: zeros at $\sin z = 0$, i.*e.* $z = n\pi$. Simple poles at $\cos z = 0$, $z = \left(n + \frac{1}{2}\right)\pi$.

b) $\frac{z-2}{z^2} \sin\left(\frac{1}{1-z}\right)$: zeros at $z = 2$, $z = 1 - \frac{1}{n\pi}$, $z = \infty$; double pole at $z = 0$, essential singularity at $z = 1$.

c) $e^{1/z}$: essential singularity at $z = 0$.

d) $\tan\left(\frac{1}{z}\right)$: zeros and poles at inverses of part a) — essential singularity at $z = 0$.

e) $z^{2/3}$: branch point at $z = 0$; branch line extending to $z = \infty$ along any smooth curve.

18.16 The equation of the ellipse is

$$\frac{l}{r} = 1 - \varepsilon \cos\theta$$

hence the area is given by

$$A = \frac{1}{2} \int_0^{2\pi} d\theta \, r^2(\theta) = \frac{1}{2} l^2 \int_0^{2\pi} \frac{d\theta}{(1 - \varepsilon \cos\theta)^2}.$$

The easiest way to get the answer is to factor out ε and consider the integral

$$I(a) = \int_0^{2\pi} \frac{d\theta}{a - \cos\theta}$$

since

$$A \equiv -l^2 \varepsilon^{-2} \left. \frac{dI}{da} \right|_{a = \frac{1}{\varepsilon}}.$$

Clearly,

$$I(a) = -2i \oint_{|z|=1} \frac{dz}{2az - 1 - z^2}$$

This we evaluate using the calculus of residues to get

$$I(a) = 2\pi\left(a^2 - 1\right)^{-1/2}$$

or

$$A = 2\pi l^2\left(1 - \varepsilon^2\right)^{-3/2}.$$

18.17 We have already done something like this in class. We want to show that

$$I(\alpha) \overset{df}{=} \int_0^\infty \frac{t\,\sin(\alpha t)\,dt}{t^2 + 1} = \pi\,e^{-\alpha}.$$

(In fact this is a factor 2 too large!) We use the fact that $t\sin(\alpha t)$ is odd to write

$$I(\alpha) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{t\,\sin(\alpha t)\,dt}{t^2 + 1} \equiv \frac{1}{2}\,\mathrm{Im}\int_{-\infty}^{\infty} \frac{t\,e^{i\alpha t}\,dt}{t^2 + 1}$$

and then integrate around the contour shown to the right. The only included pole is at $+i$, and the integral on the large semicircle vanishes as R^{-1} by Jordan's Lemma, so we get

$$I(\alpha) = \mathrm{Im}\left(\pi i\,e^{-\alpha}\,\frac{i}{2i}\right) = \frac{\pi}{2}\,e^{-\alpha}.$$

Note this differs by a factor 2 from the answer given in the book. (Riley, 1st ed.)