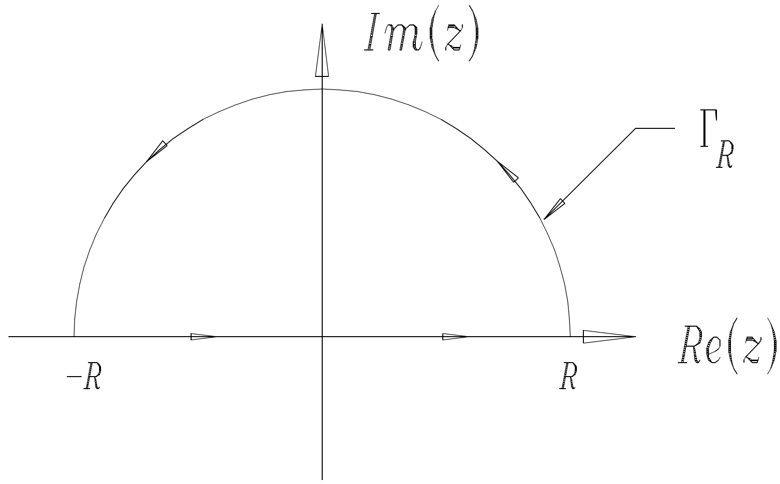
**Riley,** ***et al.*** **p. 670**

18.18 $I = \int_0^\infty dx \frac{\cos mx}{4x^4 + 5x^2 + 1}$ ; integrand is even so write (contour shown below)

$$\oint_\Gamma dz \frac{e^{imz}}{4z^4 + 5z^2 + 1} = 2I + \lim_{R\to\infty} O\left(\tfrac{1}{R^3}\right)$$

$$= 2\pi i\left(\sum residues\right) = 2\pi i\left(\frac{e^{-m/2}}{2i\left(1-\frac{1}{4}\right)} + \frac{e^{-m}}{(-4+1)2i}\right)$$

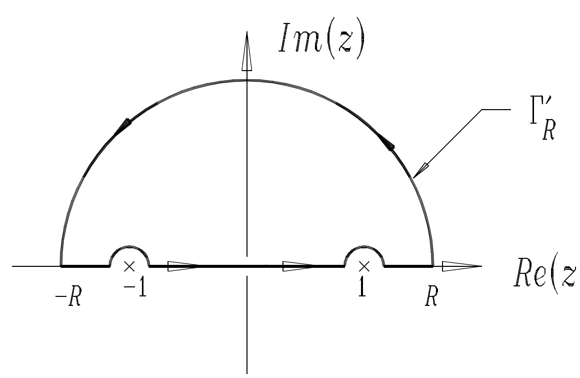
$$= \frac{\pi}{3}\left(4e^{-m/2} - e^{-m}\right)$$




18.19 We evaluate the integral

$$\oint_\Gamma dz \frac{e^{iz}}{z^2 - 1}$$

on the contour shown at the right. Clearly we may neglect the contribution from the large semicircle. The part along the real axis is the principal value integral we want (the part from sin $x$ vanishes because it is odd), and the total integral vanishes. The contributions from the small semicircles are



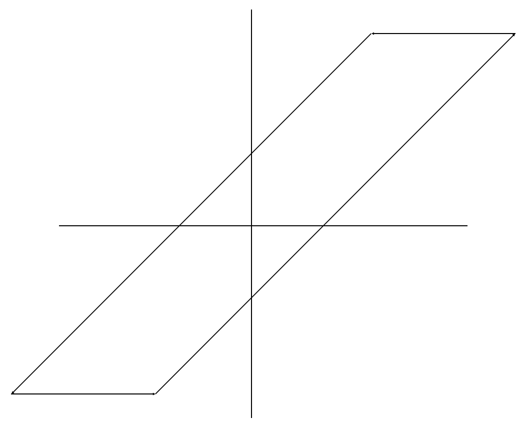
$$i\varepsilon\int_{\pi}^{0} d\theta\, e^{i\theta}\,\frac{e^{-i}}{-2\varepsilon e^{i\theta}} + i\varepsilon\int_{\pi}^{0} d\theta\, e^{i\theta}\,\frac{e^{i}}{2\varepsilon e^{i\theta}} = \frac{-\pi\, i}{2}\left(e^{i} - e^{-i}\right) = \pi\sin(1)$$

and the total integral vanishes. The result is

$$P\int_{-\infty}^{\infty} dx\,\frac{\cos(x/a)}{x^2-1} = -\frac{\pi}{a}\sin(1)\,.$$

That is, the answer in the book has the wrong algebraic sign.

18.20 a) The only pole in the parallelogram is at $z = 0$. Its residue is $\lim_{z\to 0}\frac{z}{\sin\pi z} = \frac{1}{\pi}$ so the contribution to the integral, by the residue theorem, is $\frac{2\pi i}{\pi} = 2i$ .



b) The lower line parallel to the real axis is parameterized by

$$z = -Re^{i\pi/4} + x$$

where $x$ runs from –0.5 to 0.5. The leading term is of order $\exp(i\pi z^2) \underset{R\to\infty}{\longrightarrow} \exp(-\pi R^2)$ and this vanishes in the limit. The same is true on the upper parallel line.

c) If we let $z = \pm\frac{1}{2} + se^{i\pi/4}$ the sum of the two integrals on the diagonals becomes

$$\lambda\int_{-R}^{R} ds\left(\frac{e^{i\pi(s\lambda+1/2)^2}}{\sin\pi(s\lambda+1/2)} - \frac{e^{i\pi(s\lambda-1/2)^2}}{\sin\pi(s\lambda-1/2)}\right)$$

where $\lambda = e^{i\pi/4}$. The above integral is easily seen to reduce to $2i\int_{-R}^{R} ds\ e^{-\pi s^2}$ so taking the limit as $R\to\infty$ we get

$$\int_{-\infty}^{\infty} ds\ e^{-\pi s^2} = 1\,.$$

**Non-Riley problems:**

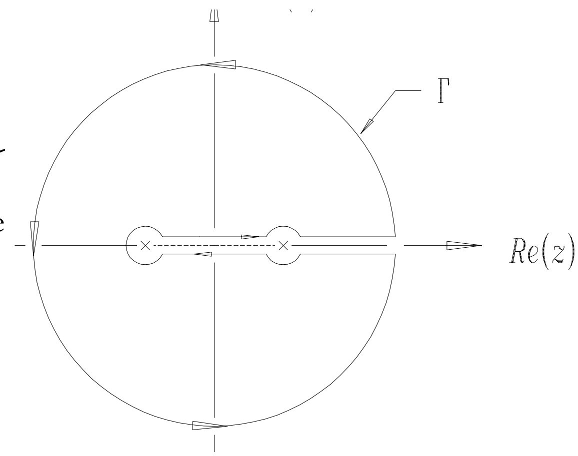
1. Integrate $e^{1/z}$, $\sin\left(\frac{1}{z}\right)$ and $\cos\left(\frac{1}{z}\right)$ around the contour $|z| = 1$.

   This can be done two ways: write the Laurent series for each function (clearly they are uniformly convergent for all $z$) and note that only the term in $\frac{1}{z}$ survives and gives $2\pi i$. This term is present in $e^{1/z}$ and $\sin\left(\frac{1}{z}\right)$ but not in $\cos\left(\frac{1}{z}\right)$. Hence the latter integral gives 0.

2. If we integrate the function $z^2\left(1 - z^2\right)^{-1/2}$ around the contour $\Gamma$ shown at the right, we see that since it includes no singularities, the integral vanishes by Cauchy's theorem. Choosing the square root to be real and positive just above the real line between –1 and +1, we see that with

$$I = \int_{-1}^{+1} dx \frac{x^2}{\sqrt{1 - x^2}}$$

   the integral around the contour gives




$$2I + \lim_{R\to\infty} i\int_0^{2\pi} d\theta \left(R\, e^{i\theta}\right)^3 \left[1 - R^2 e^{2i\theta}\right]^{-1/2} = 0 .$$

   The integral around the circle can be rewritten

$$i\int_0^{2\pi} d\theta \left(R\, e^{i\theta}\right)^3 \left(1 - R^2 e^{2i\theta}\right)^{-1/2} = \frac{i}{-i}\int_0^{2\pi} d\theta \left(R\, e^{i\theta}\right)^2 \left(1 - R^{-2} e^{-2i\theta}\right)^{-1/2}$$

$$= -\int_0^{2\pi} d\theta \left(R\, e^{i\theta}\right)^2 \left[1 + \frac{1}{2} R^{-2} e^{-2i\theta} + \frac{3}{8} R^{-4} e^{-4i\theta} + \dots\right] = -\pi .$$

   Hence $I = \frac{\pi}{2}$.

3. We are to evaluate the integral

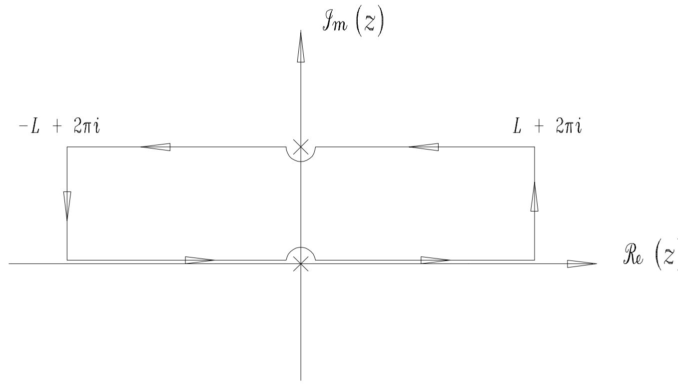
$$I(\lambda) = \wp\int_{-\infty}^{\infty} \frac{dx\, e^{\lambda x}}{e^x - 1}, \quad 0 < \lambda < 1$$

   in closed form, and relate it to the infinite series representation

$$I(\lambda) = -\frac{1}{\lambda} + 2\lambda \sum_{n=1}^{\infty} \frac{1}{n^2 - \lambda^2}$$

We use the contour shown to the right, and integrate the function

$$\frac{e^{\lambda z}}{e^z - 1};$$




clearly the integrand has no poles or other singularities within the contour, so the integral around it vanishes. We see that the integral along the real axis gives us the Cauchy principal value integral $I(\lambda)$ as well as the contribution from the small semicircle that avoids the pole at $z = 0$ . The contribution from the line $z = x + 2\pi i$ is $-e^{2\pi i\lambda}\, I(\lambda)$ and of course we have to include the small semicircle that avoids the pole at $z = 2\pi i$ . Finally, the contributions from the vertical segments of the contour vanish in the limit $L \to \infty$ . Adding these together we have

$$\left(1 - e^{2\pi i\lambda}\right) I(\lambda) + \lim_{\varepsilon \to 0} \left[ i\varepsilon \int_{\pi}^{0} \frac{e^{\lambda\varepsilon e^{i\theta}} e^{i\theta} d\theta}{e^{\varepsilon e^{i\theta}} - 1} + i\varepsilon e^{2\pi i\lambda} \int_{0}^{-\pi} \frac{e^{\lambda\varepsilon e^{i\theta}} e^{i\theta} d\theta}{e^{\varepsilon e^{i\theta} + 2\pi i} - 1} \right] = 0$$

or

$$I(\lambda) = -\pi \cot(\pi\lambda) .$$

To see that the series representation is correct, we note that

$$\wp \int_{-\infty}^{\infty} \frac{dx\, e^{\lambda x}}{e^x - 1} = \lim_{\varepsilon \to 0} \left[ \int_{-\infty}^{-\varepsilon} \frac{dx\, e^{\lambda x}}{e^x - 1} + \int_{\varepsilon}^{\infty} \frac{dx\, e^{\lambda x}}{e^x - 1} \right] = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} dx \left( \frac{e^{-\lambda x}}{e^{-x} - 1} + \frac{e^{(\lambda - 1)x}}{1 - e^{-x}} \right)$$

which can be rewritten

$$\lim_{\varepsilon \to 0} \left( -\frac{1}{\lambda} - \sum_{n=1}^{\infty} \frac{e^{-(\lambda+n)\varepsilon}}{n + \lambda} + \sum_{n=1}^{\infty} \frac{e^{-(n-\lambda)\varepsilon}}{n - \lambda} \right) = -\frac{1}{\lambda} + 2\lambda \sum_{n=1}^{\infty} \frac{1}{n^2 - \lambda^2} + \lim_{\varepsilon \to 0} O(-\varepsilon \ln \varepsilon)$$

Now we expand both expressions for $I(\lambda)$ in powers of $\lambda$ and compare term by term: we have

$$-\pi \cot \pi\lambda = -\frac{1}{\lambda} \left( 1 - \frac{(\pi\lambda)^2}{2} + \frac{(\pi\lambda)^4}{24} + \dots \right) \left( 1 - \frac{(\pi\lambda)^2}{6} + \frac{(\pi\lambda)^4}{120} + \dots \right)^{-1}$$

$$-\frac{1}{\lambda} + \lambda \frac{\pi^2}{3} + \lambda^3 \frac{\pi^4}{45} + \dots = -\frac{1}{\lambda} + 2\lambda \sum_{1}^{\infty} \frac{1}{n^2} + 2\lambda^3 \sum_{1}^{\infty} \frac{1}{n^4} + \dots$$

from which we derive

$$\sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\sum_{1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

and so forth.