PHYS 725 HW #1. Due 13 September 2001

1. Riley 5.4 Part (a):

   Rescale:

   $$x = a\xi, \quad y = b\eta\,;$$

   then

   $$A = ab \int_{-1}^{1} d\eta \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} d\xi = 4ab \int_0^1 d\eta \sqrt{1-\eta^2}\,.$$

   Now let

   $$\eta = \sin\theta, \quad d\eta = \cos\theta\, d\theta$$

   and we see

   $$A = 4ab \int_0^{\pi/2} d\theta\ \cos^2\theta = ab \int_0^{\pi} d\theta\ (1+\cos\theta) = \pi ab\,.$$

   Part (b): If we consider a slice of thickness dz at position z we see that

   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{z^2}{c^2}\,,$$

   so that the volume of the slice is

   $$dV = \pi ab \left(\sqrt{1-\frac{z^2}{c^2}}\right)^2 dz\,.$$

   We are to integrate this from $z = -c$ to $z = c$; hence with $z = c\zeta$,

   $$V = 2\pi abc \int_0^1 d\varsigma \left(1 - \varsigma^2\right) = \frac{4\pi}{3} abc\,.$$

2. Riley 5.5

Part (a):

$$\Psi_1 = \frac{1}{\sqrt{\pi}} a^{-3/2} e^{-r/a}$$

$$\int d^3r \, |\Psi_1|^2 = \frac{4\pi}{\pi} a^{-3} \int_0^\infty dr \, r^2 e^{-2r/a} = \frac{1}{2} \int_0^\infty d\rho \, \rho^2 e^{-\rho} = 1 \, .$$

$$\Psi_2 = -\sqrt{\frac{1}{64\pi}} \, (a)^{-5/2} \, r e^{-r/2a} \sin\theta e^{i\varphi}$$

$$\int d^3r \, |\Psi_2|^2 = \frac{(8\pi/3)}{64\pi} a^{-5} \int_0^\infty dr \, r^4 e^{-r/a} = \frac{4!}{24} = 1 \, .$$

Part (b):

$$\begin{aligned} p_x &= -\frac{1}{8\pi} a^{-8/2} \int d^3r \, e^{-r/a} q r^2 e^{-r/2a} \sin^2\theta \cos\varphi \, e^{i\varphi} \\ &= -\frac{qa}{8\pi} \int_0^\infty d\rho \, \rho^4 e^{-3\rho/2} \int_0^\pi d\theta \, \sin^3\theta \int_0^{2\pi} d\varphi \, \cos^2\varphi \\ &= -\frac{qa}{8\pi} \, \pi \, \frac{4}{3} \, \left(\frac{2}{3}\right)^5 4 = -qa \left(2^7 \big/ 3^5\right) \, . \end{aligned}$$

3. Riley 5.6

Part (a):

$$\begin{aligned} I &= \mu \iint dx \, dy \, \left(x^2 + y^2\right) = 2\pi\mu \int_0^R dr \, r \left(x^2 + y^2\right) \\ &= 2\pi\mu \int_0^R dr \, r^3 = \frac{2\pi\mu R^4}{4} = \frac{1}{2} R^2 \left(\pi\mu R^2\right) = \frac{1}{2} M R^2 \end{aligned}$$

Part (b):

$$
\begin{aligned}
I &= \mu \iint dx\, dy\, x^2 = \mu \int_0^R dr\, r^3 \int_0^{2\pi} d\theta\, \cos^2\theta \\
&= \frac{\pi\mu R^4}{4} = \frac{1}{4} R^2 \left(\pi\mu R^2\right) = \frac{1}{4} M R^2 \,.
\end{aligned}
$$

4. Riley 5.7

   Note that the problem is wrongly stated. *Moral homily # 42c states that you must be wary of formulas found in texts or journal articles. This is an example.*

   The correct definition of the moment of inertia about an axis is

$$
I_z \stackrel{df}{=} \int dm\, (\hat{z} \times \vec{r})^2
$$

   where $\hat{z}$ is any axis of rotation you choose. So for the cylinder we have the moment about the symmetry axis is the same as for a disk, namely

$$
I_z \stackrel{df}{=} \int dm\, (\hat{z} \times \vec{r})^2 = \frac{1}{2} M a^2 \,.
$$

   The moment of inertia about an axis $\perp$ the symmetry axis and passing through the center of mass can be obtained as

$$
\begin{aligned}
I_x &= \int dm\, (\hat{x} \times \vec{r})^2 = \int dm\, \left(y^2 + z^2\right) \\
&= \rho \int_{-b}^{b} dz \left(\pi a^2 z^2 + \frac{\pi}{4} a^4\right) = M \left(\frac{a^2}{4} + \frac{b^2}{3}\right) \,.
\end{aligned}
$$

   Notice that Riley's answer is wrong.

5. Riley 3.4

(a) (Comparison test)

$$\left|\sum_{1}^{\infty} \frac{2\sin n\theta}{n\,(n+1)}\right| \leq \sum_{1}^{\infty} \frac{2}{n\,(n+1)} = 2\,.$$

(b) (Integral test)

$$\sum_{1}^{\infty} \frac{2}{n^2} \leq 2 + 2\int_{1}^{\infty} \frac{dx}{x^2} = 4\,.$$

(c) (Integral test)

$$\sum_{1}^{N} \frac{1}{2n^{1/2}} \geq \int_{1}^{N} \frac{dx}{2\sqrt{x}} = \left(\sqrt{N} - 1\right) \to \infty\,.$$

(d) (Weierstrass's theorem)

$$\sum_{2}^{\infty} \frac{(-1)^n \sqrt{n^2+1}}{n \ln n} \equiv \sum_{2}^{\infty} \frac{(-1)^n \sqrt{1 + 1/n^2}}{\ln n}$$

and

$$\frac{\frac{\sqrt{1+1/(n+1)^2}}{\ln(n+1)}}{\frac{\sqrt{1+1/n^2}}{\ln n}} = \frac{\ln n}{\ln(n+1)}\sqrt{\frac{1+1/(n+1)^2}{1+1/n^2}} < 1\,.$$

That is, we have a series of terms that decrease monotonically in magnitude and alternate in sign, hence it converges.

(e) (Ratio test) Manifestly, since $n! \to n^n$ it dominates n^p for any finite p. Thus the ratio of successive terms is

$$\frac{(n+1)^p}{n^p}\,\frac{1}{n+1} \to \frac{1}{n+1} + \frac{p}{n\,(n+1)} + \ldots \to 0 < 1\,.$$

Or in other words, the *radius of convergence* is infinite.

6. Riley 3.5

(a) From the ratio test we see that the radius of convergence is 1 so that the series converges for all $|x| < 1$. Of course we can sum the series using

$$f(x) = \sum_1^\infty \frac{x^n}{n+1}$$

$$\frac{d}{dx}(xf(x)) = \sum_1^\infty x^n = \frac{1}{1-x} - 1$$

$$f(x) = -\frac{\ln(1-x)}{x} - 1\,,$$

and see that it goes bad at $x = +1$. Even for $x = -1$ the series converges by Weierstrass's theorem, but is diverges for $x < -1$. The answer is therefore $-1 \leq x < 1$.

(b) $$\sum_1^\infty (\sin x)^n = \frac{\sin x}{1 - \sin x}\,.$$

Clearly the series converges for $-\pi/2 < x < \pi/2$, diverges for $x = \pi/2$, and is indefinite for $x = -\pi/2$.

(c) The terms are all positive. They are increasing or constant for $x \geq 0$ so we must have $x < 0$. However, the integral test gives

$$\sum_1^\infty n^x \geq \int_1^\infty du\, u^x + \text{constant} = \left(\frac{u^{x+1}}{x+1}\right)\Bigg|_1^\infty + \text{constant}$$

hence we must have $x < -1$ for this to be finite. The case $x = -1$ is excluded by the fact that this is the harmonic series, known to be divergent.

(d) $$\sum_1^\infty e^{nx} = \frac{e^x}{1-e^x}\,;$$

we see immediately that for $x \geq 0$ the series is divergent, but for $x < 0$, $e^x < 1$ so the series manifestly converges for $x < 0$.

(e) Clearly we must assume $x < 0$ since otherwise the terms are increasing. In fact, by the comparison test we can see that $x < -1$ since for $x = -1$ the harmonic series is a lower bound. Let us therefore apply the integral test to the case $x < -1$. We have

$$\begin{aligned}
\sum_2^\infty (\ln n)^x &\geq \int_2^\infty du\ (\ln u)^x \\
&= \left[u\ (\ln u)^x\right]\Big|_2^\infty - x\int_2^\infty du\ (\ln u)^{x-1}\,; \\
&> \lim_{N\to\infty} \frac{N}{(\ln N)^{|x|}} + \text{constant} \\
&\sim \lim_{N\to\infty} \left(\frac{N^{1/|x|}}{\ln N}\right)^{|x|} = \infty\,.
\end{aligned}$$

However, for any positive power—say $\alpha = 1/|x|$—it is true that for large enough N, $N^\alpha > \ln N$ so we know that the lower bound diverges. In other words, there is no real value of x for which the series converges.

(f) Riley 3.6: Manifestly,

$$B = A\,(1-r)\sum_0^\infty \left(re^{i\varphi}\right)^n = \frac{A\,(1-r)}{1-re^{i\varphi}}\,,$$

therefore

$$|B|^2 = \left|\frac{A\,(1-r)}{1-re^{i\varphi}}\right|^2 = |A|^2\,\frac{(1-r)^2}{1-2r\cos\varphi + r^2}\,.$$

7. Riley 3.8: Let $u = 0.5(a-b)$ and $v = 0.5(a+b)$. Then $a = u+v$ and $b = v-u$. Assuming u is small in magnitue, we can write

$$\begin{aligned}
\ln\frac{a}{b} &= \ln\left(\frac{v+u}{v-u}\right) = \ln\left(\frac{1+u/v}{1-u/v}\right) \\
&\approx 2\frac{u}{v} + \frac{2}{3}\left(\frac{u}{v}\right)^3 + \ldots = 2\left(\frac{a-b}{a+b}\right) + \frac{2}{3}\left(\frac{a-b}{a+b}\right)^3
\end{aligned}$$

8. Riley 3.12:

$$\begin{aligned} V_i &= \frac{q_i}{R}\left(\sum_{j=1}^{\infty}\frac{q_j}{j}+\sum_{j=-1}^{-\infty}\frac{q_j}{|j|}\right)=\frac{2e^2}{R}\mathrm{sgn}\,(q_i)\sum_{j=1}^{\infty}\frac{(-1)^j}{j} \\ &= -\frac{2e^2}{R}\mathrm{sgn}\,(q_i)\ln 2\,. \end{aligned}$$

9. Riley 3.13:

$$\begin{aligned} \coth x-\frac{1}{x} &= \frac{1}{x}\left(\frac{1+\frac{x^2}{2}+\frac{x^4}{24}+\dots}{1+\frac{x^2}{6}+\frac{x^4}{120}+\dots}-1\right) \\ &= \frac{1}{x}\left(\frac{1+\frac{x^2}{2}+\frac{x^4}{24}+\dots-1-\frac{x^2}{6}-\frac{x^4}{120}-\dots}{1+\frac{x^2}{6}+\frac{x^4}{120}+\dots}\right) \\ &= x\left(\frac{\frac{1}{2}-\frac{1}{6}+x^2\left(\frac{1}{24}-\frac{1}{120}\right)+\dots}{1+\frac{x^2}{6}+\dots}\right) \\ &\approx \frac{x}{3}\left(1+\frac{x^2}{10}\right)\left(1-\frac{x^2}{6}\right)\approx\frac{x}{3}-\frac{x^3}{45}\,. \end{aligned}$$