PHYS 725 HW #4. Due 15 November 2001

1. Riley 12.3:

\[ R \frac{dq}{dt} + \frac{q}{C} = V(t); \]

The solution is obtained with the integrating factor \( \exp \left( \frac{t}{RC} \right) \), giving

\[ q(t) = e^{-t/RC} \left( \frac{1}{R} \int_0^t ds V(s) e^{s/RC} + q(0) \right). \]

With \( q(0) = 0 \) and \( V(t) = V_0 \sin(\omega t) \) we thus have

\[
q(t) = e^{-t/RC} \frac{V_0}{R} \left[ \frac{1}{i\omega RC} \left( e^{i\omega t} - e^{-t/RC} \right) \right] \\
= CV_0 \left[ \frac{\sin(\omega t - \tan^{-1}(\omega RC))}{\sqrt{1 + (\omega RC)^2}} + \frac{\omega RC e^{-t/RC}}{\sqrt{1 + (\omega RC)^2}} \right].
\]

We can see this is right by comparing the behavior at small \( t \)—we should get

\[ q(t) \approx \omega V_0 \frac{t^2}{2R}. \]

2. Riley 12.4:

The equation

\[ (y - x) \frac{dy}{dx} + 2x + 3y = 0 \]

is homogeneous of degree 1, so substituting \( y(x) = xv(x) \) we find

\[ (v - 1) \frac{dv}{dx} + 1 + (v + 1)^2 = 0 \]

or with \( v = u - 1 \),

\[
\int \frac{(u - 2)}{1 + u^2} du = \frac{1}{2} \ln \left( 1 + u^2 \right) - 2 \tan^{-1} u = -\int dx = -x + A.
\]
3. Riley 12.9:
The equation
\[ \sin x \frac{dy}{dx} + 2y \cos x = 1 \]
can be reduced to a quadrature by the standard integrating factor,
\[ f(x) = \exp \left[ 2 \int \frac{\cos t}{\sin t} dt \right] = \exp (2 \ln(\sin x)) = \sin^2 x ; \]
applying this we have
\[ \sin^2 x \frac{dy}{dx} + 2y \cos x \sin x = \frac{d}{dx} \left( y \sin^2 x \right) = \sin x \]
or
\[ y \sin^2 x = - \cos x + 1 = \frac{\sin^2 x}{1 + \cos x}, \]
\[ y = \frac{1}{1 + \cos x} \]
where we have applied the boundary condition \( y(\pi/2) = 1 \) to determine the constant in the solution.

4. Riley 13.6:
Use the method of variation of parameters to find the general solutions of
\[ (a) \quad \frac{d^2 y}{dx^2} - y = x^n \]

**Solution:** The independent solutions of the homogeneous equation are \( e^x \) and \( e^{-x} \) so we let
\[ y(x) = \alpha (x) e^x + \beta (x) e^{-x}, \]
with the subsidiary condition
\[ e^x \alpha' + e^{-x} \beta' = 0. \]
Then differentiating twice and applying the subsidiary condition, we have
\[
\frac{d^2y}{dx^2} = \alpha e^x + \beta e^{-x} + e^x \alpha' - e^{-x} \beta',
\]
or
\[
e^x \alpha' - e^{-x} \beta' = x^n.
\]
Thus
\[
y(x) = Ae^x + Be^{-x} + \int_0^x dt t^n \sinh (x - t).
\]
(b) \( \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + y = 2xe^x \)

**Solution:** The independent solutions of the homogeneous equation are \( e^x \) and \( xe^x \) so let
\[
y(x) = \alpha(x) e^x + \beta(x) xe^x
\]
and find \( \beta = x^2, \alpha = -2x^3/3. \)

5. Riley 13.7:
The Green’s function is
\[
G(x, t) = \frac{y_2(x)y_1(t)}{w(x)} \theta(x - t) + \frac{y_1(x)y_2(t)}{w(x)} \theta(t - x)
\]
where \( y_2(0) \neq 0, y_2(\pi) = 0, y_1(0) = 0, y_1(\pi) \neq 0. \) Since the solutions of the homogeneous equation that satisfy these criteria are \( y_1(x) = \sin x/2, y_2(x) = \cos x/2, \) and since the Wronskian is
\[
w(x) = \frac{dy_2}{dx} y_1 - \frac{dy_1}{dx} y_2 = \frac{1}{2} \sin^2 \left( \frac{x}{2} \right) - \frac{1}{2} \cos^2 \left( \frac{x}{2} \right) = -\frac{1}{2}
\]
we have
\[
G(x, t) = -2 \cos (x/2) \sin (t/2) \theta(x - t) - 2 \cos (t/2) \sin (x/2) \theta(t - x).
\]

6. Riley 14.4 Part (a):
\[
zy'' - 2y' + zy = 0
\]
so let

\[ y(z) = \sum_{n=0}^{\infty} a_n z^{n+\alpha}; \]

the indicial equation is

\[ \alpha (\alpha - 1) - 2\alpha = 0, \]
or \( \alpha = 0, 3. \)

We get a 2-term recursion relation

\[ (\alpha + n + 2) (\alpha + n - 1) a_{n+2} + a_n = 0. \]

With \( \alpha = 0 \) and \( a_1 = 0 \), the terms are

\[ z^0 + \frac{1}{2} z^2 - \frac{1}{4} \frac{3}{2} z^4 + \frac{1}{6} \frac{3}{4} \frac{5}{2} z^6 - \frac{1}{8} \frac{5}{6} \frac{3}{4} \frac{7}{2} z^8 + \ldots \]

which we rewrite as

\[ \frac{(-1)^0}{0!} z^0 + \frac{1}{2!} z^2 - \frac{3}{4!} \frac{5}{2} z^4 + \frac{5}{6!} \frac{7}{2} z^6 - \frac{7}{8!} \frac{9}{2} z^8 + \ldots \]

\[ = \sum_{n=0}^{\infty} \frac{(2n - 1) (-1)^{n+1}}{(2n)!} z^{2n} = y_2(z). \]

With \( \alpha = 3 \) we get, similarly, the terms

\[ z^3 - \frac{1}{2 \cdot 5} z^5 + \frac{1}{2 \cdot 4 \cdot 5 \cdot 7} z^7 - + \ldots \]

\[ = \frac{3 \cdot 2}{3!} z^3 - \frac{3 \cdot 4 \cdot 6}{5!} z^5 + \frac{3 \cdot 6}{7!} z^7 - + \ldots \]

\[ = 3 \sum_{n=1}^{\infty} \frac{2n (-1)^{n+1}}{(2n+1)!} z^{2n+1} = y_1(z). \]
Part (b): If we expand the sinusoidal functions in power series we get

\[
\sin z = z \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} - \frac{z^{2n+1}}{(2n)!}
\]

\[
= \sum_{n=0}^{\infty} (-1)^n \frac{(2n + 1 - 1) z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{2n z^{2n+1}}{(2n+1)!},
\]

which is \(y_1(z)\) within the requisite factor 3.

To get the other solution using the Wronskian method we write

\[
y_2(z) = u(z) y_1(z)
\]

so that

\[
u'(z) = A \frac{z^2}{[y_1(z)]^2}.
\]

Integrating we find

\[
y_2(z) = u(z) y_1(z) = Ay_1(z) \int^z \frac{t^2 dt}{[y_1(t)]^2} = Ay_1(z) \int^z \frac{t^2 dt}{(\sin t - t \cos t)^2},
\]

or using the hint to perform the integral by parts,

\[
y_2(z) = A \left( z \sin z + \cos z \right).
\]

Expanding we recover the series

\[
- \frac{(-1)^0}{0!} z^0 + \frac{1}{2!} z^2 - \frac{3}{4!} z^4 + \frac{5}{6!} z^6 - \frac{7}{8!} z^8 + \ldots
\]

which we identify with \(y_2\), within a multiplicative factor.

Part (c): Calculating the Wronskian we get

\[
(z \sin z + \cos z)' (\sin z - z \cos z) - (z \sin z + \cos z) (\sin z - z \cos z)'
\]

\[
= (z \cos z) (\sin z - z \cos z) - (z \sin z + \cos z) (z \sin z) = -z^2 \neq 0.
\]
7. Riley 14.5

The equation is \( y'' - 2zy' - 2y = 0 \); the power series solution about \( z = 0 \) is

\[
y(z) = \sum_{n=0}^{\infty} c_n z^n
\]

leading to the recursion relation \( c_{n+2} = 2c_n/(n+2) \); with \( c_0 = 1 \) and \( c_1 = 0 \) we get

\[
y(z) = \exp \left( z^2 \right).
\]

We can get a second solution using \( y_1(z) = \exp \left( z^2 \right) v(z) \) which gives \( v'' + 2zv' = 0 \) or

\[
v(z) = A \int_0^z dt \exp \left( -t^2 \right) + v(0)
\]

which, with \( A = 1 \) and \( v(0) = 0 \) gives

\[
y_1(z) = \int_0^z dt \exp \left( z^2 - t^2 \right).
\]

But this must be the power-series solution obtained with \( c_0 = 0, c_1 = 1 \) which is

\[
y_1(z) = z \sum_{n=0}^{\infty} \frac{(2z^2)^n}{(2n+1)!!} \equiv \int_0^z dt \exp \left( z^2 - t^2 \right),
\]

where \( (2n+1)!! \equiv (2n+1) \times (2n-1) \times \ldots \times (1) \).

8. Riley 14.8

The differential equation for the Hermite polynomials is

\[
H_n'' - 2zH_n' + 2nH_n = 0;
\]
if we define the generating function

\[
G(z, t) = \sum_{n=0}^{\infty} \frac{H_n(z) t^n}{n!}
\]

then the differential equation may be multiplied by \(t^n/n!\) and summed to get

\[
\frac{\partial^2 G}{\partial z^2} - 2z \frac{\partial G}{\partial z} + 2t \frac{\partial G}{\partial t} = 0.
\]

Since we are given the solution,

\[
G(z, t) = \sum_{n=0}^{\infty} \frac{H_n(z) t^n}{n!} = \exp\left(2zt - t^2\right),
\]

we can differentiate with respect to \(z\) to get

\[
\sum_{n=0}^{\infty} H_n'(z) \frac{t^n}{n!} = 2t \exp\left(2zt - t^2\right) = 2 \sum_{n=0}^{\infty} H_n(z) \frac{t^{n+1}}{n!}.
\]

Comparing like powers of \(t\) we see that

\[
\frac{dH_n}{dz} = 2nH_{n-1}.
\]

We can also differentiate \(G\) with respect to \(t\) to get

\[
\frac{d}{dt} \exp\left(2zt - t^2\right) = 2(z - t) \sum_{n=0}^{\infty} H_n(z) \frac{t^n}{n!} = \sum_{n=1}^{\infty} H_n(z) \frac{nt^{n-1}}{n!}
\]

or

\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} \left[2zH_n(z) - 2nH_{n-1}(z) - H_{n+1}(z)\right] = 0.
\]

Comparing coefficients of \(t^n\) we get the desired result,

\[
H_{n+1} - 2zH_n + 2nH_{n-1} = 0.
\]
9. Riley 14.9

Clearly

\[ G(z, t) = \exp \left(2zt - t^2\right) \equiv \exp \left(2zt - t^2 - z^2 + z^2\right) = \exp \left(z^2\right) \exp \left((z - t)^2\right) \]

so that

\[ \exp \left(-z^2\right) H_n(z) = \left. \frac{\partial^n}{\partial t^n} \exp \left(- (z - t)^2\right) \right|_{t=0} \]

\[ \equiv \left( \frac{-\partial}{\partial z} \right)^n \exp \left(- (z - t)^2\right) \bigg|_{t=0} = \left( \frac{-\partial}{\partial z} \right)^n \exp \left(-z^2\right) \]

or

\[ H_n(z) = \exp \left(z^2\right) \left( \frac{-\partial}{\partial z} \right)^n \exp \left(-z^2\right) . \]

10. Non-riley problem:

The driven, damped oscillator is defined by

\[ \ddot{x} + \gamma \dot{x} + \omega^2 x = \frac{f(t)}{m} = Q(t) . \]

Using the operator method (or Laplace transform, or variation of parameters) we find

\[ x(t) = \int_0^t ds K(t - s) Q(s) + x_0(t) \]

where

\[ K(t - s) = \frac{1}{\Omega} e^{-\gamma(t-s)/2} \sin \left[ \Omega (t - s) \right] , \]

\[ \Omega^2 = \omega^2 - \frac{\gamma^2}{4} , \]

and where \( x_0(t) \) is any solution of the homogeneous equation. Similarly, by direct differentiation or any other method we find

\[ \dot{x}(t) = \int_0^t ds \lambda (t - s) Q(s) + \dot{x}_0(t) - \frac{\gamma}{2} x(t) , \]
where
\[
\Lambda(t-s) = e^{-\gamma(t-s)/2} \cos \Omega(t-s).
\]

We now assume \(Q(t)\) is a random function with the ensemble averages characteristic of Gaussian white noise:
\[
\langle Q(t) \rangle = 0
\]
\[
\langle Q(t)Q(s) \rangle = \frac{\sigma^2}{m^2} \delta(t-s).
\]

Then we can find the expected values and variances of \(x(t)\) and \(\dot{x}(t)\):
\[
\langle x(t) \rangle = x_0(t),
\]
\[
\langle \dot{x}(t) \rangle = \dot{x}_0(t) - \frac{\gamma}{2} \langle x(t) \rangle.
\]

\[
\langle (x(t) - \langle x(t) \rangle)^2 \rangle = \frac{\sigma^2}{m^2} \int_0^t ds \ [K(t-s)]^2
\rightarrow_{t \rightarrow \infty} \frac{\sigma^2}{2m^2 \Omega^2 \gamma} \left( 1 - \frac{\gamma^2}{\gamma^2 + 4 \Omega^2} \right) = \frac{\sigma^2}{2m^2 \omega^2 \gamma}
\]

\[
\langle (\dot{x}(t) - \langle \dot{x}(t) \rangle)^2 \rangle = \frac{\sigma^2}{m^2} \left( \int_0^t ds \ [\Lambda(s)]^2 - \gamma \int_0^t ds \ \Lambda(s) K(s) + \frac{\gamma^2}{4} \int_0^t ds \ [K(s)]^2 \right)
\rightarrow_{t \rightarrow \infty} \frac{\sigma^2}{2m^2 \gamma} \left( 1 + \frac{\gamma^2}{4 \omega^2} + \frac{\gamma^2}{4 \omega^2} - \frac{2 \gamma^2}{4 \omega^2} \right) = \frac{\sigma^2}{2m^2 \gamma}
\]

Thus, for large \(t\), after the system has settled down, the ensemble average of the (fluctuational) energy of a harmonic oscillator driven by noise is
\[
\langle H \rangle = \frac{m}{2} \left[ \langle (\dot{x}(t) - \langle \dot{x}(t) \rangle)^2 \rangle + \omega^2 \langle (x(t) - \langle x(t) \rangle)^2 \rangle \right] = \frac{\sigma^2}{m \gamma}.
\]
Note that this energy is independent of the oscillator frequency, as long as the oscillator is underdamped.

This result is exactly twice the ensemble-averaged kinetic energy which, in the limit that the particle is unbound, is expected to be

\[
\frac{\sigma^2}{2m\gamma} = \frac{kT}{2},
\]

that is, the equilibrium thermal energy of an oscillator in a thermal bath at absolute temperature \( T \) is \( kT \).