PHYS 725 HW #4. Due 15 November 2001

1. Riley 12.3:

$$R\frac{dq}{dt} + \frac{q}{C} = V(t)\,;$$

The solution is obtained with the integrating factor $\exp(t/RC)$, giving

$$q(t) = e^{-t/RC}\left(\frac{1}{R}\int_0^t dsV(s)\,e^{s/RC} + q(0)\right)\,.$$

With $q(0) = 0$ and $V(t) = V_0\sin(\omega t)$ we thus have

$$\begin{aligned} q(t) &= e^{-t/RC}\frac{V_0}{R}\mathrm{Im}\left[\int_0^t ds\,e^{i\omega s}e^{s/RC}\right] \\ &= CV_0\mathrm{Im}\left[\frac{1}{i\omega RC+1}\left(e^{i\omega t} - e^{-t/RC}\right)\right] \\ &= \frac{CV_0}{\sqrt{1+(\omega RC)^2}}\left[\sin\left(\omega t - \tan^{-1}(\omega RC)\right) + \frac{\omega RCe^{-t/RC}}{\sqrt{1+(\omega RC)^2}}\right]\,. \end{aligned}$$

We can see this is right by comparing the behavior at small t—we should get

$$q(t) \approx \omega V_0\frac{t^2}{2R}\,.$$

2. Riley 12.4:
The equation

$$(y-x)\frac{dy}{dx} + 2x + 3y = 0$$

is homogeneous of degree 1, so substituting $y(x) = x\,v(x)$ we find

$$(v-1)\frac{dv}{dx} + 1 + (v+1)^2 = 0$$

or with $v = u - 1$,

$$\int\frac{(u-2)\,du}{1+u^2} = \frac{1}{2}\ln\left(1+u^2\right) - 2\tan^{-1}u = -\int dx = -x + A\,.$$

3. Riley 12.9:
The equation

$$\sin x \frac{dy}{dx} + 2y\cos x = 1$$

can be reduced to a quadrature by the standard integrating factor,

$$f(x) = \exp\left[2\int^x dt \frac{\cos t}{\sin t}\right] = \exp\left(2\ln(\sin x)\right) = \sin^2 x\,;$$

applying this we have

$$\sin^2 x \frac{dy}{dx} + 2y\cos x \sin x \equiv \frac{d}{dx}\left(y\sin^2 x\right) = \sin x$$

or

$$y\sin^2 x = -\cos x + 1 = \frac{\sin^2 x}{1+\cos x}\,,$$

$$y = \frac{1}{1+\cos x}$$

where we have applied the boundary condition $y\,(\pi/2) = 1$ to determine the constant in the solution.

4. Riley 13.6:
Use the method of variation of parameters to find the general solutions of

   (a) $\frac{d^2y}{dx^2} - y = x^n$

   Solution: The independent solutions of the homogeneous equation are e^x and e^{-x} so we let

   $$y(x) = \alpha\,(x)\,e^x + \beta\,(x)\,e^{-x}\,,$$

   with the subsidiary condition

   $$e^x\alpha' + e^{-x}\beta' = 0\,.$$

Then differentiating twice and applying the subsidiary condition, we have

$$\frac{d^2y}{dx^2} = \alpha e^x + \beta e^{-x} + e^x \alpha' - e^{-x}\beta' \,,$$

or

$$e^x \alpha' - e^{-x}\beta' = x^n \,.$$

Thus

$$y(x) = Ae^x + Be^{-x} + \int_0^x dt\, t^n \sinh{(x-t)} \,.$$

(b) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 2xe^x$

Solution: The independent solutions of the homogeneous equation are e^x and xe^x so let

$$y\,(x) = \alpha\,(x)\, e^x + \beta\,(x)\, xe^x$$

and find $\beta = x^2$, $\alpha = -2x^3/3$.

5. Riley 13.7:
The Green's function is

$$G(x,t) = \frac{y_2(x)y_1(t)}{w(x)}\theta(x-t) + \frac{y_1(x)y_2(t)}{w(x)}\theta(t-x)$$

where $y_2(0) \neq 0$, $y_2(\pi) = 0$, $y_1(0) = 0$, $y_(\pi) \neq 0$. Since the solutions of the homogeneous equation that satisfy these criteria are $y_1\,(x) = \sin{(x/2)}$, $y_2\,(x) = \cos{(x/2)}$, and since the Wronskian is

$$w(x) = \frac{dy_2}{dx}y_1 - \frac{dy_1}{dx}y_2 = -\frac{1}{2}\sin^2\left(\frac{x}{2}\right) - \frac{1}{2}\cos^2\left(\frac{x}{2}\right) = -\frac{1}{2}$$

we have

$$G(x,t) = -2\cos{(x/2)}\sin{(t/2)}\,\theta(x-t) - 2\cos{(t/2)}\sin{(x/2)}\,\theta(t-x)\,.$$

6. Riley 14.4 Part (a):

$$zy'' - 2y' + zy = 0$$

so let

$$y(z) = \sum_{n=0}^{\infty} a_n z^{n+\alpha} ;$$

the indicial equation is

$$\alpha(\alpha - 1) - 2\alpha = 0 ,$$

or $\alpha = 0$, 3.

We get a 2-term recursion relation

$$(\alpha + n + 2)(\alpha + n - 1)\, a_{n+2} + a_n = 0 .$$

With $\alpha = 0$ and $a_1 = 0$, the terms are

$$z^0 + \frac{1}{2} z^2 - \frac{}{4 \cdot 2} z^4 + \frac{1}{6 \cdot 3 \cdot 4 \cdot 2} z^6 - \frac{1}{8 \cdot 5 \cdot 6 \cdot 3 \cdot 4 \cdot 2} z^8 + \dots$$

which we rewrite as

$$\begin{aligned} -\frac{(-1)}{0!} z^0 \quad &+ \quad \frac{1}{2!} z^2 - \frac{3}{4!} z^4 + \frac{5}{6!} z^6 - \frac{7}{8!} z^8 + \dots \\ &= \quad \sum_{n=0}^{\infty} \frac{(2n-1)(-1)^{n+1}}{(2n)!} z^{2n} = y_2(z) . \end{aligned}$$

With $\alpha = 3$ we get, similarly, the terms

$$\begin{aligned} z^3 \quad &- \quad \frac{1}{2 \cdot 5} z^5 + \frac{1}{2 \cdot 4 \cdot 5 \cdot 7} z^7 - + \dots \\ &= \quad \frac{3 \cdot 2}{3!} z^3 - \frac{3 \cdot 4}{5!} z^5 + \frac{3 \cdot 6}{7!} z^7 - + \dots \\ &= \quad 3 \sum_{n=1}^{\infty} \frac{2n\,(-1)^{n+1}}{(2n+1)!} z^{2n+1} = y_1(z) \ . \end{aligned}$$

Part (b): If we expand the sinusoidal functions in power series we get

$$\begin{aligned}\sin z \quad - \quad z\cos z &= \sum_{n=0}^{\infty}(-1)^n\left[\frac{z^{2n+1}}{(2n+1)!} \quad - \frac{z^{2n+1}}{(2n)!}\right] \\ &= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{(2n+1-1)\,z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{2nz^{2n+1}}{(2n+1)!},\end{aligned}$$

which is $y_1(z)$ within the requisite factor 3.
To get the other solution using the Wronskian method we write

$$y_2(z) = u(z)\; y_1(z)$$

so that

$$u'(z) = A\frac{z^2}{[y_1(z)]^2}\,.$$

Integrating we find

$$y_2(z) = u(z)\; y_1(z) = Ay_1(z)\int^z \frac{t^2 dt}{[y_1(t)]^2} = Ay_1(z)\int^z \frac{t^2 dt}{(\sin t - t\cos t)^2}\,,$$

or using the hint to perform the integral by parts,

$$y_2(z) = A\,(z\sin z + \cos z)\ .$$

Expanding we recover the series

$$-\frac{(-1)}{0!}z^0 + \frac{1}{2!}z^2 - \frac{3}{4!}z^4 + \frac{5}{6!}z^6 - \frac{7}{8!}z^8 + \ldots$$

which we identify with y_2, within a multiplicative factor.
Part (c): Calculating the Wronskian we get

$$\begin{aligned}&(z\sin z + \cos z)'\;(\sin z - z\cos z) - (z\sin z + \cos z)\;(\sin z - z\cos z)' \\ &\quad = (z\cos z)\;(\sin z - z\cos z) - (z\sin z + \cos z)\;(z\sin z) = -z^2 \not\equiv 0\,.\end{aligned}$$

7. Riley 14.5

The equation is $y'' - 2zy' - 2y = 0$; the power series solution about $z = 0$ is

$$y(z) = \sum_{n=0}^{\infty} c_n z^n$$

leading to the recursion relation $c_{n+2} = 2c_n/(n+2)$; with $c_0 = 1$ and $c_1 = 0$ we get

$$y(z) = \exp\left(z^2\right) .$$

We can get a second solution using $y_1(z) = \exp\left(z^2\right) v(z)$ which gives $v'' + 2zv' = 0$ or

$$v(z) = A \int_0^z dt \, \exp\left(-t^2\right) + v(0)$$

which, with $A = 1$ and $v(0) = 0$ gives

$$y_1(z) = \int_0^z dt \, \exp\left(z^2 - t^2\right) .$$

But this must be the power-series solution obtained with $c_0 = 0$, $c_1 = 1$ which is

$$y_1(z) = z \sum_{n=0}^{\infty} \left(2z^2\right)^n \frac{1}{(2n+1)!!} \equiv \int_0^z dt \, \exp\left(z^2 - t^2\right) ,$$

where $(2n+1)!! \stackrel{df}{=} (2n+1) \times (2n-1) \times \ldots \times (1)$.

8. Riley 14.8

The differential equation for the Hermite polynomials is

$$H_n'' - 2zH_n' + 2nH_n = 0 ;$$

if we define the generating function

$$G(z,t) \stackrel{df}{=} \sum_{n=0}^{\infty} H_n(z) \frac{t^n}{n!}$$

then the differential equation may be multiplied by $t^n/n!$ and summed to get

$$\frac{\partial^2 G}{\partial z^2} - 2z\frac{\partial G}{\partial z} + 2t\frac{\partial G}{\partial t} = 0\,.$$

Since we are given the solution,

$$G(z,t) \stackrel{df}{=} \sum_{n=0}^{\infty} H_n(z) \frac{t^n}{n!} = \exp\left(2zt - t^2\right)\,,$$

we can differentiate with respect to z to get

$$\sum_{n=0}^{\infty} H_n'(z) \frac{t^n}{n!} = 2t \exp\left(2zt - t^2\right) = 2\sum_{n=0}^{\infty} H_n(z) \frac{t^{n+1}}{n!}\,.$$

Comparing like powers of t we see that

$$\frac{dH_n}{dz} = 2nH_{n-1}\,.$$

We can also differentiate G with respect to t to get

$$\frac{d}{dt}\exp\left(2zt - t^2\right) = 2(z-t)\sum_{n=0}^{\infty} H_n(z)\frac{t^n}{n!} = \sum_{n=1}^{\infty} H_n(z)\frac{nt^{n-1}}{n!}$$

or

$$\sum_{n=0}^{\infty} \frac{t^n}{n}\left[2zH_n(z) - 2nH_{n-1}(z) - H_{n+1}(z)\right] = 0\,.$$

Comparing coefficients of t^n we get the desired result,

$$H_{n+1} - 2zH_n + 2nH_{n-1} = 0\,.$$

9. Riley 14.9

   Clearly

$$G(z,t) = \exp\left(2zt - t^2\right) \equiv \exp\left(2zt - t^2 - z^2 + z^2\right) = \exp\left(z^2\right)\exp\left((z-t)^2\right)$$

   so that

$$\begin{aligned} \exp\left(-z^2\right) H_n(z) &= \left.\frac{\partial^n}{\partial t^n} \exp\left(-(z-t)^2\right)\right|_{t=0} \\ &\equiv \left.\left(\frac{-\partial}{\partial z}\right)^n \exp\left(-(z-t)^2\right)\right|_{t=0} = \left(\frac{-\partial}{\partial z}\right)^n \exp\left(-z^2\right) \end{aligned}$$

   or

$$H_n(z) = \exp\left(z^2\right)\left(\frac{-\partial}{\partial z}\right)^n \exp\left(-z^2\right) .$$

10. Non-riley problem:
    The driven, damped oscillator is defined by

$$\ddot{x} + \gamma \dot{x} + \omega^2 x = \frac{f(t)}{m} = Q(t) .$$

    Using the operator method (or Laplace transform, or variation of parameters) we find

$$x(t) = \int_0^t ds K(t-s) Q(s) + x_0(t)$$

    where

$$K(t-s) = \frac{1}{\Omega} e^{-\gamma(t-s)/2} \sin\left[\Omega(t-s)\right] ,$$

$$\Omega^2 = \omega^2 - \frac{\gamma^2}{4} ,$$

    and where $x_0(t)$ is any solution of the homogeneous equation. Similarly, by direct differentiation or any other method we find

$$\dot{x}(t) = \int_0^t ds \Lambda(t-s) Q(s) + \dot{x}_0(t) - \frac{\gamma}{2} x(t) ,$$

where

$$\Lambda (t-s) = e^{-\gamma(t-s)/2} \cos\left[\Omega (t-s)\right] .$$

We now assume $Q(t)$ is a random function with the ensemble averages characteristic of Gaussian white noise:

$$\langle Q(t) \rangle = 0$$

$$\langle Q(t)Q(s) \rangle = \frac{\sigma^2}{m^2} \delta (t-s) .$$

Then we can find the expected values and variances of $x(t)$ and $\dot{x}(t)$:

$$\langle x(t) \rangle = x_0(t) ,$$

$$\langle \dot{x}(t) \rangle = \dot{x}_0(t) - \frac{\gamma}{2} \langle x(t) \rangle$$

$$\begin{aligned} \left\langle (x(t) - \langle x(t) \rangle)^2 \right\rangle &= \frac{\sigma^2}{m^2} \int_0^t ds \, \left[K(t-s)\right]^2 \\ &\underset{t\to\infty}{\rightarrow} \frac{\sigma^2}{2m^2\Omega^2\gamma} \left(1 - \frac{\gamma^2}{\gamma^2 + 4\Omega^2}\right) = \frac{\sigma^2}{2m^2\omega^2\gamma} \end{aligned}$$

$$\begin{aligned} &\left\langle (\dot{x}(t) - \langle \dot{x}(t) \rangle)^2 \right\rangle = \\ &\quad \frac{\sigma^2}{m^2} \left(\int_0^t ds \, \left[\Lambda(s)\right]^2 - \gamma \int_0^t ds \, \Lambda(s) \, K(s) + \frac{\gamma^2}{4} \int_0^t ds \, \left[K(s)\right]^2 \right) \\ &\quad \underset{t\to\infty}{\rightarrow} \frac{\sigma^2}{2m^2\gamma} \left(1 + \frac{\gamma^2}{4\omega^2} + \frac{\gamma^2}{4\omega^2} - \frac{2\gamma^2}{4\omega^2}\right) = \frac{\sigma^2}{2m^2\gamma} \end{aligned}$$

Thus, for large t, after the system has settled down, the ensemble average of the (fluctuational) energy of a harmonic oscillator driven by noise is

$$\langle H \rangle = \frac{m}{2} \left[\left\langle (\dot{x}(t) - \langle \dot{x}(t) \rangle)^2 \right\rangle + \omega^2 \left\langle (x(t) - \langle x(t) \rangle)^2 \right\rangle \right] = \frac{\sigma^2}{m\gamma} .$$

Note that this energy is independent of the oscillator frequency, as long as the oscillator is underdamped.

This result is exactly twice the ensemble-averaged kinetic energy which, in the limit that the particle is unbound, is expected to be

$$\frac{\sigma^2}{2m\gamma} = \frac{kT}{2}\,;$$

that is, the equilibrium thermal energy of an oscillator in a thermal bath at absolute temperature T is kT.