PHYS 725 Midterm Examination

This is a pledged take-home exam. Answer all 8 questions. It is open book, and there is no time limit. However it must be turned in Tuesday, November 6, 2001, **in class**. You might find it valuable, as practice for the final, to study the questions, then try to write the solutions within 3 hours, without further consulting notes or books.

1. Evaluate the integral

$$\int_0^\infty dx \frac{\ln x}{1+x^3}$$

in closed form using Cauchy's Theorem. **Hint:** use the contour shown in the notes for the integral

$$\int_0^\infty dx \frac{1}{1+x^3} \,,$$

and ask yourself what function has a discontinuity (across the positive real axis) proportional to $\ln x$.

2. Evaluate the integral

$$I = \int_0^\infty dx \frac{\sinh \alpha x}{\sinh \pi x} \,.$$

For what (real) range of α is it finite?

3. Evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{(a + b\cos\theta)^2} \,.$$

Hint: find a way to express the above integral in terms of the simpler integral

$$\int_0^{2\pi} \frac{d\theta}{(a + b\cos\theta)} \,.$$

4. The Laplace transform of a function $y(x)$ is defined by

$$\tilde{y}(\lambda) \stackrel{df}{=} \int_0^\infty dx\, y(x)\, e^{-\lambda x},$$

assuming the integral is well-defined.

(a) What is the Laplace transform of $Dy \stackrel{df}{=} dy/dx$, the first derivative of y?

(b) Laplace transform the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = xe^{-2x}$$

and thereby determine $\tilde{y}$ in terms of $y(0)$ and $y'(0)$.

(c) The inverse Laplace transform of a function (which gives back the original function when its Laplace transform is known) is defined by

$$y(x) = \int_{\gamma - i\infty}^{\gamma + i\infty} d\lambda\, \tilde{y}(\lambda)\, e^{\lambda x},$$

where $\gamma > 0$. Use this to determine the solution of the above differential equation when $y(0) = 0$ and $y'(0) = 1$.

5. Evaluate the sum

$$S = \sum_{n=1}^{\infty} \frac{1}{n^4 + n^2}$$

by contour integration.

6. Characterize the location(s) and type(s) of the singularities of each of the following functions

(a) $f(z) = 3/(z^2 + z^4)$.

(b) $f(z) = \sinh(1/z)$

(c) $f(z) = \int_1^z \frac{dt}{t}$

(d) $$f(z) = \int_0^\infty dt\, e^{-t^3(1+z)}.$$

Hint: a change of integration variable might help!

7. The 0'th order Bessel function has the infinite series expansion

$$J_0(z) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left(-z^2/4\right)^n .$$

(a) For what values of z does the series converge? (Justify your answer using the convergence tests discussed in class.)

(b) Evaluate the integral

$$\lim_{R\to\infty} \oint_{|z|=R} dz\, z^2 J_0\left(1/\sqrt{z}\right) .$$

Justify the operations necessary to get your result.

8. Discuss the convergence of the following infinite series (that is, do they converge or diverge, and why).

(a) $$\sum_{n=1}^{\infty} \frac{1}{2n^{1/2}} ;$$

(b) $$\sum_{n=2}^{\infty} \frac{(-1)^n \sqrt{n^2+1}}{n \ln n} .$$