PHYS 725 Midterm Examination

This is a pledged take-home exam. Answer all 8 questions. It is open book, and there is no time limit. However it must be turned in Tuesday, November 6, 2001, in class. You might find it valuable, as practice for the final, to study the questions, then try to write the solutions within 3 hours, without further consulting notes or books.

1. Evaluate the integral

\[ \int_0^\infty dx \frac{\ln x}{1 + x^3} \]

in closed form using Cauchy’s Theorem. **Hint:** use the contour shown in the notes for the integral

\[ \int_0^\infty dx \frac{1}{1 + x^3} , \]

and ask yourself what function has a discontinuity (across the positive real axis) proportional to \( \ln x \).

2. Evaluate the integral

\[ I = \int_0^\infty dx \frac{\sinh \alpha x}{\sinh \pi x} . \]

For what (real) range of \( \alpha \) is it finite?

3. Evaluate the integral

\[ \int_0^{2\pi} d\theta \frac{d\theta}{(a + b \cos \theta)^2} . \]

**Hint:** find a way to express the above integral in terms of the simpler integral

\[ \int_0^{2\pi} d\theta \frac{d\theta}{(a + b \cos \theta)} . \]
4. The Laplace transform of a function \( y(x) \) is defined by 
\[
\tilde{y}(\lambda) = \int_{0}^{\infty} dx \ y(x) \ e^{-\lambda x},
\]
assuming the integral is well-defined.

(a) What is the Laplace transform of \( D y = dy/dx \), the first derivative of \( y \)?

(b) Laplace transform the differential equation 
\[
\frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + y = xe^{-2x}
\]

and thereby determine \( \tilde{y} \) in terms of \( y(0) \) and \( y'(0) \).

(c) The inverse Laplace transform of a function (which gives back the original function when its Laplace transform is known) is defined by 
\[
y(x) = \int_{\gamma-i\infty}^{\gamma+i\infty} d\lambda \ \tilde{y}(\lambda) \ e^{\lambda x},
\]
where \( \gamma > 0 \). Use this to determine the solution of the above differential equation when \( y(0) = 0 \) and \( y'(0) = 1 \).

5. Evaluate the sum
\[
S = \sum_{n=1}^{\infty} \frac{1}{n^4 + n^2}
\]
by contour integration.

6. Characterize the location(s) and type(s) of the singularities of each of the following functions

(a) \( f(z) = 3/(z^2 + z^4) \).
(b) \( f(z) = \sinh(1/z) \).
(c) \( f(z) = \int_{1}^{z} \frac{dt}{t} \).
(d) \( f(z) = \int_{0}^{\infty} dt \ e^{-t^2(1+z)}. \)

\textbf{Hint:} a change of integration variable might help!
7. The 0’th order Bessel function has the infinite series expansion

\[ J_0(z) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left( -\frac{z^2}{4} \right)^n. \]

(a) For what values of \( z \) does the series converge? (Justify your answer using the convergence tests discussed in class.)

(b) Evaluate the integral

\[ \lim_{R \to \infty} \int_{|z|=R} dz \; z^2 J_0 \left( \frac{1}{\sqrt{z}} \right). \]

Justify the operations necessary to get your result.

8. Discuss the convergence of the following infinite series (that is, do they converge or diverge, and why).

(a) \[ \sum_{n=1}^{\infty} \frac{1}{2n^{1/2}}; \]

(b) \[ \sum_{n=2}^{\infty} \frac{(-1)^n \sqrt{n^2} + 1}{n \ln n}. \]