PHYS 725 Midterm Examination

This is a pledged take-home exam. Answer all 8 questions. It is open book, and there is no time limit. However it must be turned in Tuesday, November 6, 2001, **in class**. You might find it valuable, as practice for the final, to study the questions, then try to write the solutions within 3 hours, without further consulting notes or books.

1. Evaluate the integral

$$\int_0^\infty dx \frac{\ln x}{1+x^3}$$

in closed form using Cauchy's Theorem. **Hint:** use the contour shown in the notes for the integral

$$J = \int_0^\infty dx \frac{1}{1+x^3}\,,$$

and ask yourself what function has a discontinuity (across the positive real axis) proportional to $\ln x$.

Solution:

$$\begin{aligned}\oint dz \frac{\ln^2 z}{1+z^3} &= \int_0^\infty dx \frac{\ln^2 x}{1+x^3} + \int_\infty^0 dx \frac{(\ln x + 2\pi i)^2}{1+x^3} + \lim_{R\to\infty} \mathcal{O}\left(R^{-2}\ln^2 R\right) \\ &= 2\pi i \sum residues\,.\end{aligned}$$

We see that

$$-4\pi i \int_0^\infty dx \frac{\ln x}{1+x^3} + 4\pi^2 J = 2\pi i \left[\frac{(i\pi/3)^2}{3e^{2\pi i/3}} + \frac{(i\pi)^2}{3e^{2\pi i}} + \frac{(5i\pi/3)^2}{3e^{10\pi i/3}}\right]$$

giving

$$\int_0^\infty dx \frac{\ln x}{1+x^3} = -\frac{2\pi^2}{27}\,.$$

The fact that the integral is *negative* can be verified by noting that

$$\int_0^\infty dx \frac{\ln x}{1+x^3} \equiv \int_0^1 dx \frac{1-x}{1+x^3} \ln x < 0\,.$$

2. Evaluate the integral

$$I = \int_0^\infty dx \frac{\sinh \alpha x}{\sinh \pi x} \, .$$

For what (real) range of α is it finite?

Solution: The integral is only well-defined for $|\mathrm{Re}(\alpha)| < \pi$. We have

$$I = \int_0^\infty dx \frac{\sinh \alpha x}{\sinh \pi x} = \tfrac{1}{2} \int_{-\infty}^\infty dx \frac{\sinh \alpha x}{\sinh \pi x}$$

which we can evaluate by considering the contour integral

$$\oint dz \frac{\sinh \alpha z}{\sinh \pi z}$$

around the rectangular contour whose corners are $-L+i0$, $L+i0$, $L+2i$, $-L+2i$ in the limit $L \to \infty$. There are simple poles at $z = i$ and $z = 2i$. The former is included within the contour, whereas the latter is avoided by a semicircle of radius ε running from $\theta = 2\pi$ to $\theta = \pi$. The result is (noting the contributions from the sides at $x = \pm L$ vanish in the limit $L \to \infty$)

$$2I\left(1 - \cos 2\alpha\right) - i \sin 2\alpha \, \mathcal{P} \int_{-\infty}^\infty dx \frac{\cosh \alpha z}{\sinh \pi z} = 2 \sin \alpha - \sin 2\alpha$$

or

$$I = \frac{\sin \alpha \left(1 - \cos \alpha\right)}{1 - \cos 2\alpha} \equiv \tfrac{1}{2} \tan\left(\alpha/2\right) \, .$$

3. Evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{\left(a + b \cos \theta\right)^2} \, .$$

Hint: find a way to express the above integral in terms of the simpler integral

$$\int_0^{2\pi} \frac{d\theta}{\left(a + b \cos \theta\right)} \, .$$

Solution: The integral we are looking for can be evaluated in terms of a simpler one:

$$\int_0^{2\pi} \frac{d\theta}{\left(a + b\cos\theta\right)^2} \equiv -\frac{d}{da}\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{d}{da}\oint_{|z|=1} \frac{2idz}{bz^2 + 2az + b}\,.$$

Clearly we must have $|a| > |b|$ or the integral will not be well-defined. For $a > 0$ the result is

$$\frac{2\pi}{\sqrt{a^2 - b^2}}\,,$$

whereas it changes sign for $a < 0$. Thus the integral we are to evaluate is

$$\int_0^{2\pi} \frac{d\theta}{\left(a + b\cos\theta\right)^2} = \frac{2\pi\,|a|}{\left(a^2 - b^2\right)^{3/2}}\,;$$

that is, it is positive (as it should be) for real a and b.

4. The Laplace transform of a function $y(x)$ is defined by

$$\tilde{y}\left(\lambda\right) \overset{df}{=} \int_0^{\infty} dx\, y\left(x\right) e^{-\lambda x}\,,$$

assuming the integral is well-defined.

   (a) What is the Laplace transform of $Dy \overset{df}{=} dy/dx$, the first derivative of y?

   Solution:

$$\begin{aligned}\mathcal{L}\left(y'\right) &= \int_0^{\infty} dx\, y'\left(x\right) e^{-\lambda x} = y\left(x\right) e^{-\lambda x}\Big|_0^{\infty} + \lambda \int_0^{\infty} dx\, y\left(x\right) e^{-\lambda x} \\ &= \lambda\tilde{y}\left(\lambda\right) - y_0\,.\end{aligned}$$

   (b) Laplace transform the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = xe^{-2x}$$

   and thereby determine $\tilde{y}$ in terms of $y(0)$ and $y'(0)$.

Solution: The Laplace transform of xe^{-2x} is $(2+\lambda)^{-2}$. Applying the Laplace transform twice to y'' gives

$$(\lambda+1)^2\,\tilde{y} = \frac{1}{(\lambda+2)^2} + (\lambda+2)\,y_0 + y_0'\,,$$

or

$$\tilde{y} = \frac{1}{(\lambda+1)^2}\left[\frac{1}{(\lambda+2)^2} + (\lambda+2)\,y_0 + y_0'\right]\,.$$

(c) The inverse Laplace transform of a function (which gives back the original function when its Laplace transform is known) is defined by

$$y(x) = \int_{\gamma-i\infty}^{\gamma+i\infty} d\lambda\,\tilde{y}\,(\lambda)\,e^{\lambda x}\,,$$

where $\gamma > 0$. Use this to determine the solution of the above differential equation when $y(0) = 0$ and $y'(0) = 1$.

Solution: We can calculate the inverse Laplace transform of the above function by contour integration [complete a closed contour by adding a large semicircle in the left half of the complex plane—manifestly the integral will vanish on this semicircle at least as fast as R^{-1} (by Jordan's Lemma), so it only adds a convenient form of 0 to the integral we need to evaluate]. Since the only singularities are poles at $\lambda = -2$ and $\lambda = -1$, we may use the residue theorem. Or we may note that the inverse transform of a simple pole, $(\lambda+a)^{-1}$ is e^{-ax} and that of a double pole is xe^{-ax}. The function

$$\tilde{y} = \frac{1}{(\lambda+1)^2}\left[\frac{1}{(\lambda+2)^2} + (\lambda+2)\,y_0 + y_0'\right]$$

can be rewritten in the form

$$\frac{A}{(\lambda+1)^2} + \frac{B}{\lambda+1} + \frac{C}{(\lambda+2)^2} + \frac{D}{\lambda+2}$$

where a little reflection reveals $A = 1 + y_0 + y_0'$, $B = y_0 - 2$, $C = 1$ and $D = 2$.

5. Evaluate the sum

$$S = \sum_{n=1}^{\infty} \frac{1}{n^4+n^2}$$

by contour integration.

Solution: We consider the integral around the circle $|z| = N + \frac{1}{2}$ where N is a positive integer, of the function

$$\frac{\pi \cot(\pi z)}{z^2 (1 + z^2)} .$$

The integral clearly vanishes like N^{-3} for large N. Since the function $\pi \cot(\pi z)$ has simple poles of residue 1 at the positive and negative integers (and also at $z = 0$), and since the function $[z^2 (1 + z^2)]^{-1}$ has poles at $z = 0$ and $z = \pm i$, we obtain

$$\begin{aligned} \lim_{N \to \infty} \frac{1}{2\pi i} \oint_{|z|=N+\frac{1}{2}} \frac{\pi \cot(\pi z)}{z^2 (1 + z^2)} dz &= 2 \sum_{n=1}^{\infty} \frac{1}{n^2 (1 + n^2)} \\ &+ res(0) + res(i) + res(-i) = 0 . \end{aligned}$$

Hence we see that

$$\sum_{1}^{\infty} \frac{1}{n^2 (1 + n^2)} = \frac{1}{2} + \frac{\pi^2}{6} - \frac{\pi}{2} \coth \pi .$$

6. Characterize the location(s) and type(s) of the singularities of each of the following functions

   (a) $f(z) = 3/(z^2 + z^4)$.
   Solution: Double pole at $z = 0$, simple poles at $z = \pm i$.

   (b) $f(z) = \sinh(1/z)$
   Solution: The Laurent series about $z = 0$ is non-terminating. It is also convergent for any value of $z \neq 0$. Hence this is an isolated essential singularity at $z = 0$.

   (c) $f(z) = \int_1^z \frac{dt}{t}$
   Solution: The integral is just $\ln z$ hence the singularity is a branch point at $z = 0$. The complex plane must be cut from $z = 0$ to ∞ to obtain a single-valued function.

(d) $$f(z) = \int_0^\infty dt\, e^{-t^3(1+z)}\,.$$

Hint: a change of integration variable might help!

Solution: If we change to $u = t\sqrt[3]{1+z}$ the integral becomes

$$\frac{1}{\sqrt[3]{1+z}} \int_0^\infty du\, e^{-u^3} = \frac{\text{constant}}{\sqrt[3]{1+z}}\,.$$

Thus the function has a branch point at $z = -1$.

7. The 0'th order Bessel function has the infinite series expansion

$$J_0(z) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left(-z^2/4\right)^n\,.$$

(a) For what values of z does the series converge? (Justify your answer using the convergence tests discussed in class.)

Solution: Clearly the series is absolutely convergent for any $|z| < \infty$. We can use the ratio or root test to verify this. Or alternatively, the radius of convergence is

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left(\frac{(n+1)!}{n!}\right)^2 = \lim_{n\to\infty} (n+1)^2 = \infty\,.$$

(b) Evaluate the integral

$$\lim_{R\to\infty} \oint_{|z|=R} dz\, z^2 J_0\left(1/\sqrt{z}\right)\,.$$

Justify the operations necessary to get your result.

Solution: Since $J_0(w)$ is actually a function of w^2, $J_0(1/\sqrt{z})$ has no branch cut. It is therefore an analytic function for all $|z| > 0$, and has an isolated essential singularity at $z = 0$. If we integrate the series around the contour $|z| = R$ we see that the integral of a given term is bounded above by a constant times R^{3-n}. Therefore there is no problem in integrating the series term by term (because of the infinite radius of convergence!). The terms with $n = 0$, 1, and 2—proportional respectively to R^3, R^2 and R—have coefficients involving the integrals

$$\int_0^{2\pi} d\theta e^{i(3-n)\theta} = 0, \quad n < 3\,.$$

The terms with $n > 3$ fall to zero with increasing R so it doesn't matter that their coefficients vanish identically. The only term that survives is the one with $n = 3$, for which we get

$$2\pi i \, \frac{(-1)^3}{(3!)^2 \, 4^3} = -\frac{\pi i}{1152} \, .$$

8. Discuss the convergence of the following infinite series (that is, do they converge or diverge, and why).

(a) $$\sum_{n=1}^{\infty} \frac{1}{2n^{1/2}} \, ;$$

Solution: Manifestly the series diverges, as the integral test demonstrates. That is, the partial sums diverge as $\mathcal{O}\left(N^{1/2}\right)$.

(b) $$\sum_{n=2}^{\infty} \frac{(-1)^n \sqrt{n^2+1}}{n \ln n} \, .$$

Solution: We see that

$$\frac{d}{dn}\left(\frac{\sqrt{1+\frac{1}{n^2}}}{\log n}\right) = -\left(\frac{\sqrt{1+\frac{1}{n^2}}}{n\,(\log n)^2} + \frac{1}{n^2\sqrt{1+n^2}\,\log n}\right) < 0$$

so the terms decrease monotonically in magnitude and alternate in sign. Hence, by Weierstrass's Theorem the series converges, but not absolutely (since the terms decrease only as $(\log n)^{-1}$).