PHYS 725 Midterm Examination

This is a pledged take-home exam. Answer all 8 questions. It is open book, and there is no time limit. However it must be turned in Tuesday, November 6, 2001, in class. You might find it valuable, as practice for the final, to study the questions, then try to write the solutions within 3 hours, without further consulting notes or books.

1. Evaluate the integral

   \[ \int_0^\infty dx \frac{\ln x}{1 + x^3} \]

   in closed form using Cauchy’s Theorem. Hint: use the contour shown in the notes for the integral

   \[ J = \int_0^\infty dx \frac{1}{1 + x^3}, \]

   and ask yourself what function has a discontinuity (across the positive real axis) proportional to \( \ln x \).

   Solution:

   \[
   \oint dz \frac{\ln^2 z}{1 + z^3} = \int_0^\infty dx \frac{\ln^2 x}{1 + x^3} + \int_0^\infty dx \frac{(\ln x + 2\pi i)^2}{1 + x^3} + \lim_{R \to \infty} \mathcal{O} \left( R^{-2} \ln^2 R \right)
   \]

   \[
   = 2\pi i \sum \text{residues}.
   \]

   We see that

   \[
   -4\pi i \int_0^\infty dx \frac{\ln x}{1 + x^3} + 4\pi^2 J = 2\pi i \left[ \frac{(i\pi/3)^2}{3e^{2\pi i/3}} + \frac{(i\pi)^2}{3e^{2\pi i}} + \frac{(5i\pi/3)^2}{3e^{10\pi i/3}} \right]
   \]

   giving

   \[
   \int_0^\infty dx \frac{\ln x}{1 + x^3} = -\frac{2\pi^2}{27}.
   \]

   The fact that the integral is negative can be verified by noting that

   \[
   \int_0^\infty dx \frac{\ln x}{1 + x^3} \equiv \int_0^1 dx \frac{1 - x}{1 + x^3} \ln x < 0.
   \]
2. Evaluate the integral

\[ I = \int_0^\infty dx \frac{\sinh \alpha x}{\sinh \pi x}. \]

For what (real) range of \( \alpha \) is it finite?

**Solution:** The integral is only well-defined for \(|\text{Re}(\alpha)| < \pi\). We have

\[ I = \int_0^\infty dx \frac{\sinh \alpha x}{\sinh \pi x} = \frac{1}{2} \int_{-\infty}^{\infty} dx \frac{\sinh \alpha x}{\sinh \pi x} \]

which we can evaluate by considering the contour integral

\[ \oint \frac{\sinh \alpha z}{\sinh \pi z} dz \]

around the rectangular contour whose corners are \(-L+i0, L+i0, L+2i, -L+2i\) in the limit \(L \to \infty\). There are simple poles at \(z = i\) and \(z = 2i\). The former is included within the contour, whereas the latter is avoided by a semicircle of radius \(\varepsilon\) running from \(\theta = 2\pi\) to \(\theta = \pi\). The result is (noting the contributions from the sides at \(x = \pm L\) vanish in the limit \(L \to \infty\))

\[ 2I (1 - \cos 2\alpha) - i \sin 2\alpha \text{P} \int_{-\infty}^{\infty} dx \frac{\cosh \alpha x}{\sinh \pi x} = 2 \sin \alpha - \sin 2\alpha \]

or

\[ I = \frac{\sin \alpha (1 - \cos \alpha)}{1 - \cos 2\alpha} \equiv \frac{1}{2} \tan \left(\frac{\alpha}{2}\right). \]

3. Evaluate the integral

\[ \int_0^{2\pi} \frac{d\theta}{(a + b \cos \theta)^2}. \]

**Hint:** find a way to express the above integral in terms of the simpler integral

\[ \int_0^{2\pi} \frac{d\theta}{(a + b \cos \theta)}. \]
Solution: The integral we are looking for can be evaluated in terms of a simpler one:

\[
\int_0^{2\pi} \frac{d\theta}{(a + b\cos \theta)^2} \equiv -\frac{d}{da} \int_0^{2\pi} \frac{d\theta}{a + b\cos \theta} = \frac{d}{da} \int_{|z|=1} \frac{2idz}{bz^2 + 2az + b}.
\]

Clearly we must have \( |a| > |b| \) or the integral will not be well-defined. For \( a > 0 \) the result is

\[
\frac{2\pi}{\sqrt{a^2 - b^2}},
\]

whereas it changes sign for \( a < 0 \). Thus the integral we are to evaluate is

\[
\int_0^{2\pi} \frac{d\theta}{(a + b\cos \theta)^2} = \frac{2\pi |a|}{(a^2 - b^2)^{3/2}};
\]

that is, it is positive (as it should be) for real \( a \) and \( b \).

4. The Laplace transform of a function \( y(x) \) is defined by

\[
\tilde{y}(\lambda) \overset{df}{=} \int_0^\infty dx \, y(x) \, e^{-\lambda x},
\]

assuming the integral is well-defined.

(a) What is the Laplace transform of \( Dy \overset{df}{=} dy/dx \), the first derivative of \( y \)?

Solution:

\[
\mathcal{L}(y') = \int_0^\infty dx \, y'(x) \, e^{-\lambda x} = y(x) \, e^{-\lambda x}\bigg|_0^\infty + \lambda \int_0^\infty dx \, y(x) \, e^{-\lambda x} = \lambda \tilde{y}(\lambda) - y_0.
\]

(b) Laplace transform the differential equation

\[
\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = xe^{-2x}
\]

and thereby determine \( \tilde{y} \) in terms of \( y(0) \) and \( y'(0) \).
**Solution:** The Laplace transform of $x e^{-2x}$ is $(2 + \lambda)^{-2}$. Applying the Laplace transform twice to $y''$ gives

$$ (\lambda + 1)^2 \tilde{y} = \frac{1}{(\lambda + 2)^2} + (\lambda + 2) y_0 + y'_0, $$

or

$$ \tilde{y} = \left( \frac{1}{(\lambda + 1)^2} \right) \left[ \frac{1}{(\lambda + 2)^2} + (\lambda + 2) y_0 + y'_0 \right]. $$

(c) The inverse Laplace transform of a function (which gives back the original function when its Laplace transform is known) is defined by

$$ y(x) = \int_{\gamma - i\infty}^{\gamma + i\infty} d\lambda \tilde{y}(\lambda) e^{\lambda x}, $$

where $\gamma > 0$. Use this to determine the solution of the above differential equation when $y(0) = 0$ and $y'(0) = 1$.

**Solution:** We can calculate the inverse Laplace transform of the above function by contour integration [complete a closed contour by adding a large semicircle in the left half of the complex plane—manifestly the integral will vanish on this semicircle at least as fast as $R^{-1}$ (by Jordan’s Lemma), so it only adds a convenient form of 0 to the integral we need to evaluate]. Since the only singularities are poles at $\lambda = -2$ and $\lambda = -1$, we may use the residue theorem. Or we may note that the inverse transform of a simple pole, $(\lambda + a)^{-1}$ is $e^{-ax}$ and that of a double pole is $x e^{-ax}$.

The function

$$ \tilde{y} = \left( \frac{1}{(\lambda + 1)^2} \right) \left[ \frac{1}{(\lambda + 2)^2} + (\lambda + 2) y_0 + y'_0 \right] $$

can be rewritten in the form

$$ \frac{A}{(\lambda + 1)^2} + \frac{B}{\lambda + 1} + \frac{C}{(\lambda + 2)^2} + \frac{D}{\lambda + 2} $$

where a little reflection reveals $A = 1 + y_0 + y'_0$, $B = y_0 - 2$, $C = 1$ and $D = 2$.

5. Evaluate the sum

$$ S = \sum_{n=1}^{\infty} \frac{1}{n^4 + n^2} $$

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by contour integration.

**Solution:** We consider the integral around the circle \(|z| = N + \frac{1}{2}\) where \(N\) is a positive integer, of the function

\[
\frac{\pi \cot (\pi z)}{z^2 (1 + z^2)}.
\]

The integral clearly vanishes like \(N^{-3}\) for large \(N\). Since the function \(\pi \cot (\pi z)\) has simple poles of residue 1 at the positive and negative integers (and also at \(z = 0\), and since the function \([z^2 (1 + z^2)]^{-1}\) has poles at \(z = 0\) and \(z = \pm i\), we obtain

\[
\lim_{N \to \infty} \frac{1}{2\pi i} \oint_{|z|=N+\frac{1}{2}} \frac{\pi \cot (\pi z)}{z^2 (1 + z^2)} \, dz = 2 \sum_{n=1}^{\infty} \frac{1}{n^2 (1 + n^2)}
\]

\[
+ \text{res} (0) + \text{res} (i) + \text{res} (-i) = 0.
\]

Hence we see that

\[
\sum_{n=1}^{\infty} \frac{1}{n^2 (1 + n^2)} = \frac{1}{2} + \frac{\pi^2}{6} - \frac{\pi}{2} \coth \pi.
\]

6. Characterize the location(s) and type(s) of the singularities of each of the following functions

(a) \(f(z) = 3/(z^2 + z^4)\).

**Solution:** Double pole at \(z = 0\), simple poles at \(z = \pm i\).

(b) \(f(z) = \sinh (1/z)\)

**Solution:** The Laurent series about \(z = 0\) is non-terminating. It is also convergent for any value of \(z \neq 0\). Hence this is an isolated essential singularity at \(z = 0\).

(c) \(f(z) = \int_1^z \frac{dt}{t}\)

**Solution:** The integral is just \(\ln z\) hence the singularity is a branch point at \(z = 0\). The complex plane must be cut from \(z = 0\) to \(\infty\) to obtain a single-valued function.
(d) \[ f(z) = \int_0^\infty dt \, e^{-t^2(1+z) z} . \]

**Hint:** a change of integration variable might help!

**Solution:** If we change to \( u = t \sqrt{1+z} \) the integral becomes

\[ \frac{1}{\sqrt{1+z}} \int_0^\infty du \, e^{-u^2} = \text{constant} \, \frac{1}{\sqrt{1+z}} . \]

Thus the function has a branch point at \( z = -1 \).

7. The 0'th order Bessel function has the infinite series expansion

\[ J_0(z) = \sum_{n=0}^{\infty} \frac{1}{(n!)}^2 \left(-\frac{z^2}{4}\right)^n . \]

(a) For what values of \( z \) does the series converge? (Justify your answer using the convergence tests discussed in class.)

**Solution:** Clearly the series is absolutely convergent for any \( |z| < \infty \). We can use the ratio or root test to verify this. Or alternatively, the radius of convergence is

\[ R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to \infty} \left( \frac{(n+1)!}{n} \right)^2 = \lim_{n \to \infty} (n+1)^2 = \infty . \]

(b) Evaluate the integral

\[ \lim_{R \to \infty} \oint_{|z|=R} dz \, z^2 J_0 \left( \frac{1}{\sqrt{z}} \right) . \]

Justify the operations necessary to get your result.

**Solution:** Since \( J_0(w) \) is actually a function of \( w^2 \), \( J_0 \left( \frac{1}{\sqrt{z}} \right) \) has no branch cut. It is therefore an analytic function for all \( |z| > 0 \), and has an isolated essential singularity at \( z = 0 \). If we integrate the series around the contour \( |z| = R \) we see that the integral of a given term is bounded above by a constant times \( R^{3-n} \). Therefore there is no problem in integrating the series term by term (because of the infinite radius of convergence!). The terms with \( n = 0, 1, \) and \( 2 \)—proportional respectively to \( R^3, R^2 \) and \( R \)—have coefficients involving the integrals

\[ \int_0^{2\pi} d\theta \, e^{i(3-n)\theta} = 0, \quad n < 3 . \]
The terms with \( n > 3 \) fall to zero with increasing \( R \) so it doesn’t matter that their coefficients vanish identically. The only term that survives is the one with \( n = 3 \), for which we get

\[
2\pi i \frac{(-1)^3}{(3!)^2 4^3} = -\frac{\pi i}{1152}.
\]

8. Discuss the convergence of the following infinite series (that is, do they converge or diverge, and why).

(a) \[
\sum_{n=1}^{\infty} \frac{1}{2n^{1/2}};
\]

**Solution:** Manifestly the series diverges, as the integral test demonstrates. That is, the partial sums diverge as \( O\left(N^{1/2}\right) \).

(b) \[
\sum_{n=2}^{\infty} \frac{(-1)^n \sqrt{n^2 + 1}}{n \ln n}.
\]

**Solution:** We see that

\[
\frac{d}{dn} \left( \frac{\sqrt{1 + \frac{1}{n^2}}}{\log n} \right) = - \left( \frac{\sqrt{1 + \frac{1}{n^2}}}{n \left(\log n\right)^2} + \frac{1}{n^2 \sqrt{1 + \frac{1}{n^2}} \log n} \right) < 0
\]

so the terms decrease monotonically in magnitude and alternate in sign. Hence, by Weierstrass’s Theorem the series converges, but not absolutely (since the terms decrease only as \( (\log n)^{-1} \)).