

Coherent States of the Simple Harmonic Oscillator

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What is the Wave Function of a Swinging Pendulum?

Consider a macroscopic simple harmonic oscillator, and to keep things simple assume there are no interactions with the rest of the universe. We know how to describe the motion using classical mechanics: for a given initial position and momentum, classical mechanics correctly predicts the future path, as confirmed by experiments with real (admittedly not perfect) systems. But from the Hamiltonian we could also write down Schrödinger's equation, and from that predict the future behavior of the system. Since we already know the answer from classical mechanics and experiment, quantum mechanics must give us the same result in the limiting case of a large system.

It is a worthwhile exercise to see just how this happens. Evidently, we cannot simply follow the classical method of specifying the initial position and momentum—the uncertainty principle won't allow it. What we can do, though, is to take an initial state in which the position and momentum are specified *as precisely as possible*. Such a state is called a *minimum uncertainty state* (the details can be found in my earlier lecture on the Generalized Uncertainty Principle).

In fact, the *ground state* of a simple harmonic oscillator *is* a minimum uncertainty state. This is not too surprising—it's just a localized wave packet centered at the origin. The system is as close to rest as possible, having only zero-point motion. What *is* surprising is that there are excited states of the pendulum in which this ground state wave packet swings backwards and forwards indefinitely, a quantum realization of the classical system, and the wave packet is always one of minimum uncertainty. Recall that this *doesn't* happen for a *free* particle on a line—in that case, an initial minimal uncertainty wave packet spreads out because the different momentum components move at different speeds. But for the oscillator, the potential somehow keeps the wave packet together, a minimum uncertainty wave packet at all times. These remarkable quasi-classical states are called *coherent states*, and were discovered by Schrodinger himself. They are important in many quasi-classical contexts, including laser radiation.

Our task here is to construct and analyze these coherent states and to find how they relate to the usual energy eigenstates of the oscillator.

Classical Mechanics of the Simple Harmonic Oscillator

To define the notation, let us briefly recap the dynamics of the *classical* oscillator: the constant energy is

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

or

$$p^2 + (m\omega x)^2 = 2mE, \quad \omega = \sqrt{k/m}.$$

The classical motion is most simply described in *phase space*, a two-dimensional plot in the variables $(m\omega x, p)$. In this space, the point $(m\omega x, p)$ corresponding to the position and momentum of the oscillator at an instant of time moves as time progresses at constant angular speed ω in a clockwise direction around the circle of radius $\sqrt{2mE}$ centered at the origin.

(Note: phase space is usually defined in terms of the variables (x, p) , but in describing the simple harmonic oscillator, the pair $(m\omega x, p)$ are more convenient.)

This motion is elegantly described by regarding the two-dimensional phase space as a complex plane, and defining the dimensionless complex variable

$$z = \frac{m\omega x + ip}{\sqrt{2\hbar m\omega}}.$$

The time evolution in phase space is simply

$$z(t) = z_0 e^{-i\omega t}.$$

The particular choice of (quantum!) scaling factor in defining z amounts to defining the unit of energy as $\hbar\omega$, the natural unit for the oscillator: it is easy to check that if the classical energy $E = (n + \frac{1}{2})\hbar\omega$ then the dimensionless $|z|^2$ is simply the number $n + \frac{1}{2}$ (which is of course very large, so the $\frac{1}{2}$ is insignificant).

Minimum Uncertainty Wave Packets

We established in the lecture on the Generalized Uncertainty Principle that any minimum uncertainty one-dimensional wave function (so $\Delta p \cdot \Delta x = \hbar/2$) for a particle must satisfy the linear differential equation (here $\hat{p} = -i\hbar d/dx$)

$$(\hat{p} - \langle p \rangle)\psi(x) = \lambda(\hat{x} - \langle x \rangle)\psi(x)$$

where $\langle x \rangle, \langle p \rangle, \lambda$ are constants, and λ is *pure imaginary*. The equation is easy to solve: any minimum uncertainly one-dimensional wave function is a Gaussian wave packet, having expectation value of momentum $\langle p \rangle$, centered at $\langle x \rangle$ and having width $(\Delta x)^2 = -\hbar/2i\lambda$. (Δx is defined for a state $|\psi\rangle$ by $(\Delta x)^2 = \langle \psi | (x - \langle x \rangle)^2 | \psi \rangle$.)

That is to say, the minimum uncertainly solution is:

$$\psi(x) = C e^{i\langle p \rangle x / \hbar} e^{i\lambda(x - \langle x \rangle)^2 / 2\hbar} = C e^{i\langle p \rangle x / \hbar} e^{-(x - \langle x \rangle)^2 / 4(\Delta x)^2}$$

with C the normalization constant.

In fact, the simple harmonic oscillator *ground state* $\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$ is just such a minimum uncertainty state, with

$$\lambda = im\omega, \quad \langle x \rangle = \langle p \rangle = 0; \quad (\Delta x)^2 = \frac{\hbar}{2m\omega}, \quad (\Delta p)^2 = \frac{\hbar m\omega}{2}, \quad \Delta p \cdot \Delta x = \frac{\hbar}{2}.$$

Furthermore, it is easy to see that the *displaced* ground state $\psi_0(x - x_0) = C e^{-m\omega(x-x_0)^2/2\hbar}$, with $\langle x \rangle = x_0$, and writing the normalization constant $(m\omega/\pi\hbar)^{1/4} = C$, must also be a minimum uncertainty state, *with the same* $\lambda = im\omega$. (It satisfies the necessary differential equation.) Of course, in contrast to the ground state, this displaced state is no longer an eigenstate of the Hamiltonian, and will therefore change with time.

(Both these states, $\langle x \rangle = x_0$ and $\langle x \rangle = 0$, have the same spread in x -space $(\Delta x)^2 = \hbar/2m\omega$, and the same spread in p -space, the only difference in the p direction being a phase factor $e^{ip\langle x_0 \rangle/\hbar}$ for the displaced state.)

What about the higher eigenstates of the oscillator Hamiltonian? They are *not* minimally uncertain states—for the n^{th} state, $\Delta p \cdot \Delta x = n\hbar/2$, as is easily checked using

$\frac{1}{2}(\Delta p)^2/2m = \frac{1}{2}k(\Delta x)^2 \sim \frac{1}{2}n\hbar\omega$. So, if we construct a minimally uncertain higher energy state, it will *not* be an eigenstate of the Hamiltonian.

Exercise: prove $\Delta p \cdot \Delta x = n\hbar/2$ for the n^{th} energy eigenstate. (Hint: use creation and annihilation operators.)

Eigenstates of the Annihilation Operator are Minimum Uncertainty States

Notation: We'll write

$$\langle x(t=0) \rangle = x_0, \quad \langle p(t=0) \rangle = p_0.$$

We restrict our attention here to *those minimum uncertainty states having the same spatial width as the oscillator ground state*—these are what we need, and these are the ones we'll show to be eigenstates of the annihilation operator. (Actually, more general minimum uncertainty states, known as squeezed states, are also interesting and important, but we'll not consider them here.)

Suppose that at $t = 0$ the oscillator wave function is the minimum uncertainty state

$$\psi(x, t=0) = C e^{ip_0 x/\hbar} e^{i\lambda(x-x_0)^2/2\hbar} = C e^{ip_0 x/\hbar} e^{-m\omega(x-x_0)^2/2\hbar}$$

centered at $(p_0, m\omega x_0)$ in phase space (as defined above for the classical oscillator), and with $\lambda = i m \omega$ to give it the same spatial extent as the ground state.

From the preceding section, this $\psi(x, 0)$ satisfies the minimum uncertainty equation

$$(\hat{p} - p_0)\psi(x, 0) = i m \omega (\hat{x} - x_0)\psi(x, 0).$$

Rearranging this equation (and multiplying by $-i$) shows it in a different light:

$$(m\omega\hat{x} + i\hat{p})\psi(x, 0) = (m\omega x_0 + ip_0)\psi(x, 0).$$

This is an eigenvalue equation! The wave packet $\psi(x, 0)$ is an eigenstate of the operator $(m\omega\hat{x} + i\hat{p})$ with eigenvalue $(m\omega x_0 + ip_0)$. It is *not*, of course, an eigenstate of either \hat{p} or \hat{x} taken individually.

Furthermore, the operator $(m\omega\hat{x} + i\hat{p})$ is just a constant times the annihilation operator \hat{a} —recall

$$\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} + i\hat{p}).$$

Therefore, this minimally uncertain initial wave packet $\psi(x, 0)$ is an eigenstate of the annihilation operator \hat{a} , with eigenvalue $(m\omega x_0 + ip_0) / \sqrt{2\hbar m\omega}$. (By the way, it's ok for \hat{a} to have complex eigenvalues, because \hat{a} is not a Hermitian operator.)

We can now make the connection with the complex plane representation of the classical operator: the eigenvalue $(m\omega x_0 + ip_0) / \sqrt{2\hbar m\omega}$ is precisely the parameter z_0 labeling the position of the classical operator in phase space in natural dimensionless units!

That is to say, a minimum uncertainty oscillator wave packet

$$\psi(x, t = 0) = C e^{ip_0 x / \hbar} e^{-m\omega(x-x_0)^2 / 2\hbar}$$

centered at $(m\omega x_0, p_0)$ in phase space and having the same spatial extent as the ground state, is an eigenstate of the annihilation operator

$$\hat{a}\psi(x, t = 0) = z_0\psi(x, t = 0).$$

with eigenvalue the position of its center in phase space, that is,

$$z_0 = \frac{m\omega x_0 + ip_0}{\sqrt{2\hbar m\omega}}.$$

Time Development of the Minimal Wave Packet

Turning now to the *time development* of the state, it is convenient to use the ket notation

$$|\psi(x, t=0)\rangle = |x_0, p_0\rangle$$

with $|x, p\rangle$ denoting a minimum uncertainty wave packet (with the same spatial width as the ground state) having those expectation values of position and momentum.

The time development of the ket, as usual, is given by

$$|\psi(x, t)\rangle = e^{-iHt/\hbar} |x_0, p_0\rangle.$$

We shall show that $|\psi(x, t)\rangle$ *remains* an eigenstate of the annihilation operator *for all times* t : it therefore continues to be a minimum uncertainty wave packet! (And, of course, with constant spatial extent.)

The key point in establishing this is that the annihilation operator itself has a simple time development in the Heisenberg representation,

$$\hat{a}(t) = e^{iHt/\hbar} \hat{a} e^{-iHt/\hbar} = \hat{a} e^{-i\omega t}.$$

To prove this, consider the matrix elements of $\hat{a}(t)$ between any two eigenstates $|n\rangle$ of the Hamiltonian

$$H|n\rangle = (n + \frac{1}{2})\hbar\omega|n\rangle$$

so

$$\langle m|\hat{a}(t)|n\rangle = e^{i(m+\frac{1}{2})\hbar\omega t/\hbar} \langle m|\hat{a}|n\rangle e^{-i(n+\frac{1}{2})\hbar\omega t/\hbar} = \langle n-1|\hat{a}|n\rangle e^{-i\omega t}.$$

Since the only nonzero matrix elements of the annihilation operator $\langle m|\hat{a}|n\rangle$ are for $m = n - 1$, and the energy eigenstates form a complete set, this simple time dependence is true as an *operator* equation

$$\hat{a}(t) = e^{iHt/\hbar} \hat{a} e^{-iHt/\hbar} = \hat{a} e^{-i\omega t}.$$

It is now easy to prove that

$$|\psi(x, t)\rangle = e^{-iHt/\hbar} |x_0, p_0\rangle$$

is always an eigenstate of \hat{a} :

$$\begin{aligned} \hat{a}|\psi(x, t)\rangle &= \hat{a}e^{-iHt/\hbar} |x_0, p_0\rangle \\ &= e^{-iHt/\hbar} \left(e^{iHt/\hbar} \hat{a} e^{-iHt/\hbar} \right) |x_0, p_0\rangle \\ &= e^{-iHt/\hbar} e^{-i\omega t} \hat{a} |x_0, p_0\rangle \\ &= e^{-iHt/\hbar} e^{-i\omega t} (m\omega x_0 + ip_0) / \sqrt{2\hbar m\omega} |x_0, p_0\rangle \\ &= \left(e^{-i\omega t} (m\omega x_0 + ip_0) / \sqrt{2\hbar m\omega} \right) |\psi(x, t)\rangle. \end{aligned}$$

Therefore the annihilation operator, which at $t = 0$ had the eigenvalue

$$z_0 = (m\omega x_0 + ip_0) / \sqrt{2\hbar m\omega},$$

corresponding to a minimal wave packet centered at $(m\omega x_0, p_0)$ in phase space, evolves in time t to another minimal packet (because it's still an eigenstate of the annihilation operator), and writing

$$|\langle x(t)\rangle, \langle p(t)\rangle\rangle = e^{-iHt/\hbar} |x_0, p_0\rangle,$$

the new eigenvalue of \hat{a}

$$z(t) = \frac{(m\omega \langle x(t)\rangle + i \langle p(t)\rangle)}{\sqrt{2\hbar m\omega}} = \frac{(m\omega x_0 + ip_0)}{\sqrt{2\hbar m\omega}} e^{-i\omega t} = z(0) e^{-i\omega t}.$$

Therefore, the center of the wave packet in phase space follows the classical path in time. This is made explicit by equating real and imaginary parts:

$$\begin{aligned} \langle x(t)\rangle &= x_0 \cos \omega t + (p_0 / m\omega) \sin \omega t, \\ \langle p(t)\rangle &= p_0 \cos \omega t - m\omega x_0 \sin \omega t. \end{aligned}$$

So we've found Schrödinger's "best possible" quantum description of a classical oscillator.

A Remark on Notation

We have chosen to work with the original position and momentum variables, and the complex parameter expressed as a function of those variables, throughout. We could have used the dimensionless variables introduced in the lecture on the simple harmonic oscillator,

$\xi = x/b = x\sqrt{m\omega/\hbar}$, $\pi = bp/\hbar = p/\sqrt{\hbar m\omega}$, $\hat{a} = (\hat{\xi} + i\hat{\pi})/\sqrt{2}$. This would of course also give $z = (\xi + i\pi)/\sqrt{2}$, a more compact representation, but one more thing to remember.

It's also common to denote the eigenstates of \hat{a} by α , $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, very elegant, but we've used z to keep reminding ourselves that this eigenvalue, unlike most of those encountered in quantum mechanics, is a complex number. Finally, some use the dimensionless variables $X = \sqrt{2\hbar/m\omega} x$, $P = \sqrt{\hbar/2m\omega} p$, differing from ξ, π by a factor of $\sqrt{2}$. The eigenvalue equation for the annihilation operator is very neat in this notation: $\hat{a}|z\rangle = (X + iP)|z\rangle$. We've avoided it, though, because our recommended textbook, Shankar, uses X, P for the ordinary position and momentum operators.

The Translation Operator

It's worth repeating the exercise for the simple case of the oscillator initially at rest a distance x_0 from the center. This gives a neat tie-in with the *translation operator* (defined below).

Let us then take the initial state to be

$$\psi(x, 0) = C e^{-m\omega(x-x_0)^2/2\hbar} = \psi_0(x - x_0)$$

where $\psi_0(x)$ is the ground state wave function—so we've moved the packet to the right by x_0 .

Now do a Taylor series expansion (taking x_0 to be the variable!):

$$\begin{aligned} \psi_0(x - x_0) &= \psi_0(x) - x_0 \frac{d}{dx} \psi_0(x) + \frac{x_0^2}{2!} \frac{d^2}{dx^2} \psi_0(x) - \dots \\ &= e^{-x_0 \frac{d}{dx}} \psi_0(x). \end{aligned}$$

It's clear from this that the *translation operator* $e^{-x_0 \frac{d}{dx}}$ shifts the wave function a distance x_0 to the right.

Since $\hat{p} = -i\hbar d/dx$, the translation operator can also be written as $e^{-ix_0 \hat{p}/\hbar}$, and from this it can be expressed in terms of \hat{a}, \hat{a}^\dagger , since

$$\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}} (m\omega \hat{x} + i\hat{p}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar m\omega}} (m\omega \hat{x} - i\hat{p}),$$

(\hat{p}, \hat{x} being Hermitian) so

$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}} (\hat{a}^\dagger - \hat{a}).$$

Therefore the displaced ground state wave function can be written

$$\begin{aligned}\psi_0(x-x_0) &= e^{-ix_0\hat{p}/\hbar}\psi_0(x) \\ &= e^{x_0\sqrt{m\omega/2\hbar}(\hat{a}^\dagger-\hat{a})}\psi_0(x) \\ &= e^{z_0(\hat{a}^\dagger-\hat{a})}\psi_0(x)\end{aligned}$$

for real $z_0 = x_0\sqrt{m\omega/2\hbar}$, since p_0 is zero for this initial state (the wave function is real).

In the ket notation, we have established that the minimal uncertainty state centered at x_0 , and having zero expectation value for the momentum, is

$$|x_0, 0\rangle = e^{z_0(\hat{a}^\dagger-\hat{a})}|0, 0\rangle.$$

But it's not exactly obvious that this is an eigenstate of \hat{a} with eigenvalue z_0 ! (As it must be.)

It's worth seeing how to prove that just from the properties of the operators—but to do that, we need a couple of theorems concerning exponentials of operators given in the Appendix.

First, if the commutator $[A, B]$ commutes with A and B , then $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A, B]}$. This result simplifies the right hand side of the above equation, for

$$\begin{aligned}e^{z_0(\hat{a}^\dagger-\hat{a})}|0, 0\rangle &= e^{z_0\hat{a}^\dagger} e^{-z_0\hat{a}} e^{-z_0^2[\hat{a}^\dagger, \hat{a}]/2}|0, 0\rangle \\ &= e^{-z_0^2/2} e^{z_0\hat{a}^\dagger}|0, 0\rangle\end{aligned}$$

where we have used $e^{-z_0\hat{a}}|0, 0\rangle = |0, 0\rangle$.

This is simpler, but it's still not obvious that we have an eigenstate of \hat{a} : we need the commutator $[\hat{a}, e^{z_0\hat{a}^\dagger}]$.

The second theorem we need is: if the commutator of two operators $[A, B] = c$ itself commutes with A and B , then

$$[A, e^{\lambda B}] = \lambda c e^{\lambda B}.$$

(This is easily proved by expanding the exponential—see the Appendix.)

Applying this to our case,

$$[\hat{a}, e^{z_0\hat{a}^\dagger}] = z_0 e^{z_0\hat{a}^\dagger}.$$

It follows immediately that $e^{-z_0^2/2} e^{z_0 \hat{a}^\dagger} |0,0\rangle$ is indeed an eigenstate of \hat{a} with eigenvalue $z_0 = x_0 \sqrt{m\omega/2\hbar}$. (It must also be correctly normalized because the translation $|x_0,0\rangle = e^{z_0(\hat{a}^\dagger - \hat{a})} |0,0\rangle$ is a unitary operation for real z_0 .)

How do we generalize this translation operator to an arbitrary state, with nonzero $\langle x \rangle, \langle p \rangle$?

Thinking in terms of the complex parameter space z , we need to be able to move in both the x and the p directions, using both $\hat{p} = -i\hbar d/dx$ and $\hat{x} = i\hbar d/dp$. This is slightly tricky since these operators do not commute, but their commutator is just a number, so (using the theorem proved in the Appendix) this will only affect the overall normalization.

Furthermore, both \hat{p} and \hat{x} are combinations of \hat{a}, \hat{a}^\dagger , so for the generalization of $e^{-i\langle x_0 \rangle \hat{p}/\hbar}$ from real x_0 to complex z to be unitary, it must have an *antihermitian* combination of \hat{a}, \hat{a}^\dagger in the exponent—a unitary operator has the form $U = e^{iH}$, where H is Hermitian, so iH is antihermitian.

We are led to the conclusion that

$$|\langle p \rangle, \langle x \rangle\rangle = e^{(z\hat{a}^\dagger - z^*\hat{a})} |0\rangle = |z\rangle,$$

conveniently labeling the coherent state using the complex parameter z of its center in phase space. Since this generalized translation operator is unitary, the new state is automatically correctly normalized.

How Do These States Relate to the Energy Eigenstates?

The equation above suggests the possibility of representing the displaced state $|z\rangle$ in the standard energy basis $|n\rangle$. We can simplify with the same trick used for the spatial displacement case in the last section, that is, the theorem $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$ where now $A = z\hat{a}^\dagger$, $B = -z^*\hat{a}$:

$$|z\rangle = e^{z\hat{a}^\dagger - z^*\hat{a}} |0\rangle = e^{-|z|^2/2} e^{z\hat{a}^\dagger} e^{-z^*\hat{a}} |0\rangle = e^{-|z|^2/2} e^{z\hat{a}^\dagger} |0\rangle$$

using $e^{-z^*\hat{a}} |0\rangle = |0\rangle$ since $\hat{a}|0\rangle = 0$.

It is now straightforward to expand the exponential:

$$|z\rangle = e^{-|z|^2/2} e^{z\hat{a}^\dagger} |0\rangle = e^{-|z|^2/2} \left(1 + z\hat{a}^\dagger + \frac{(z\hat{a}^\dagger)^2}{2!} + \dots \right) |0\rangle$$

and recalling that the normalized energy eigenstates are

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle$$

we find

$$|z\rangle = e^{-|z|^2/2} \left(|0\rangle + z|1\rangle + \frac{z^2}{\sqrt{2!}}|2\rangle + \frac{z^3}{\sqrt{3!}}|3\rangle + \dots \right).$$

Exercise: Check that this state is correctly normalized, and is an eigenstate of \hat{a} .

Time Development of an Eigenstate of \hat{a} Using the Energy Basis

Now that we have expressed the eigenstate $|z\rangle$ as a sum over the eigenstates $|n\rangle$ of the Hamiltonian, finding its time development in this representation is straightforward.

Since $|n(t)\rangle = e^{-in\omega t} |n\rangle$,

$$|z(t)\rangle = e^{-|z_0|^2/2} \left(|0\rangle + z_0 e^{-i\omega t} |1\rangle + \frac{z_0^2 e^{-2i\omega t}}{\sqrt{2!}} |2\rangle + \frac{z_0^3 e^{-3i\omega t}}{\sqrt{3!}} |3\rangle + \dots \right)$$

which can be written

$$|z(t)\rangle = e^{-|z_0|^2/2} e^{z_0 e^{-i\omega t} \hat{a}^\dagger} |0\rangle,$$

equivalent to the result $z(t) = z_0 e^{-i\omega t}$ derived earlier.

Some Properties of the Set of Eigenstates of \hat{a}

In quantum mechanics, any physical variable is represented by a Hermitian operator. The eigenvalues are real, the eigenstates are orthogonal (or can be chosen to be so for degenerate states) and the eigenstates for a complete set, spanning the space, so any vector in the space can be represented in a unique way as a sum over these states.

The operator \hat{a} is not Hermitian. Its eigenvalues are *all the numbers in the complex plane*. The eigenstates belonging to different eigenvalues are never orthogonal, as is immediately obvious on considering the ground state and a displaced ground state. The overlap does of course decrease rapidly for states far away in phase space.

The state overlap can be computed using $|z\rangle = e^{-|z|^2/2} e^{z\hat{a}^\dagger} |0\rangle$:

$$\langle w|z\rangle = \langle 0| e^{w^* \hat{a}} e^{-|w|^2/2} e^{-|z|^2/2} e^{z\hat{a}^\dagger} |0\rangle$$

and we can then switch the operators $e^{-w^* \hat{a}}$, $e^{z \hat{a}^\dagger}$ using the theorem from the Appendix $e^B e^A = e^A e^B e^{-[A,B]}$, then since $\langle 0 | \hat{a}^\dagger = \hat{a} | 0 \rangle = 0$, we're left with

$$\langle w | z \rangle = \langle 0 | e^{w^* \hat{a}} e^{-|w|^2/2} e^{-|z|^2/2} | 0 \rangle,$$

from which

$$\left| \langle w | z \rangle \right|^2 = e^{-|w-z|^2}.$$

Finally, using $|z\rangle = e^{-|z|^2/2} \left(|0\rangle + z|1\rangle + \frac{z^2}{\sqrt{2!}}|2\rangle + \frac{z^3}{\sqrt{3!}}|3\rangle + \dots \right)$, we can construct a unit operator using the $|z\rangle$,

$$I = \iint \frac{dx dy}{\pi} |z\rangle \langle z|$$

where the integral is over the whole complex plane $z = x + iy$ (this x is not, of course, the original position x , recall for the wave function just displaced along the axis $z_0 = x_0 \sqrt{m\omega/2\hbar}$). Therefore, the $|z\rangle$ span the whole space.

Appendix: Some Exponential Operator Algebra

Suppose that the commutator of two operators A, B

$$[A, B] = c,$$

where c commutes with A and B , usually it's just a number, for instance 1 or $i\hbar$.

Then

$$\begin{aligned} [A, e^{\lambda B}] &= [A, 1 + \lambda B + (\lambda^2/2!)B^2 + (\lambda^3/3!)B^3 + \dots] \\ &= \lambda c + (\lambda^2/2!)2Bc + (\lambda^3/3!)3B^2c + \dots \\ &= \lambda c e^{\lambda B}. \end{aligned}$$

That is to say, the commutator of A with $e^{\lambda B}$ is proportional to $e^{\lambda B}$ itself.

That is reminiscent of the simple harmonic oscillator commutation relation $[H, a^\dagger] = \hbar\omega a^\dagger$ which led directly to the ladder of eigenvalues of H separated by $\hbar\omega$. Will there be a similar "ladder" of eigenstates of A in general?

Assuming A (which is a general operator) has an eigenstate $|a\rangle$ with eigenvalue a ,

$$A|a\rangle = a|a\rangle.$$

Applying $[A, e^{\lambda B}] = \lambda c e^{\lambda B}$ to the eigenstate $|a\rangle$:

$$Ae^{\lambda B}|a\rangle = e^{\lambda B}A|a\rangle + \lambda c e^{\lambda B}|a\rangle = (a + \lambda c)|a\rangle.$$

Therefore, unless it is identically zero, $e^{\lambda B}|a\rangle$ is *also* an eigenstate of A , with eigenvalue $a + \lambda c$.

We conclude that instead of a *ladder* of eigenstates, we can apparently generate a whole *continuum* of eigenstates, since λ can be set arbitrarily!

To find more operator identities, premultiply $[A, e^{\lambda B}] = \lambda c e^{\lambda B}$ by $e^{-\lambda B}$ to find:

$$\begin{aligned} e^{-\lambda B}Ae^{\lambda B} &= A + \lambda[A, B] \\ &= A + \lambda c. \end{aligned}$$

This identity is *only* true for operators A, B whose commutator c is a number. (Well, c *could* be an operator, provided it still commutes with both A and B).

Our next task is to establish the following very handy identity, which is also only true if $[A, B]$ commutes with A and B :

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A, B]}.$$

The proof (due to Glauber, given in Messiah) is as follows:

Take $f(x) = e^{Ax} e^{Bx}$,

$$\begin{aligned} df/dx &= Ae^{Ax} e^{Bx} + e^{Ax} e^{Bx} B \\ &= f(x)(e^{-Bx}Ae^{Bx} + B) \\ &= f(x)(A + x[A, B] + B). \end{aligned}$$

It is easy to check that the solution to this first-order differential equation equal to one at $x = 0$ is

$$f(x) = e^{x(A+B)} e^{\frac{1}{2}x^2[A, B]}$$

so taking $x = 1$ gives the required identity, $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A, B]}$.

It also follows that $e^B e^A = e^A e^B e^{-[A, B]}$ provided—as always—that $[A, B]$ commutes with A and B .