Quasi-Classical States of the Simple Harmonic Oscillator

(Draft Version)

Introduction: Why Look for Eigenstates of the Annihilation Operator?

Except for the ground state, the correspondence between the quantum energy eigenstates of the simple harmonic oscillator and the classical states in which a pendulum swings back and forth is not completely clear. While it is true that the high n limit of the x-space probability distribution in the energy eigenstate $|n\rangle$ approaches the time-averaged probability distribution for the classical system, the probability distribution for the quantum state is *time-independent*: it is a stationary state. On the other hand, the classical motion observed by anyone looking at a pendulum is manifestly one with a time dependent probability distribution! Our aim in this section is to construct quantum states that go in the classical limit to the "swinging pendulum" state. Evidently, these quantum states, having time-varying probability distributions, cannot be eigenstates of the Hamiltonian.

The motion of a classical oscillator is best understood in phase space. For the one-dimensional oscillator the phase space is two-dimensional, with position x along the x-axis, momentum p along the y-axis. The swinging pendulum describes a circle (in suitably scaled units) centered at the origin in the (x,p) plane, with constant angular velocity ω.

The (dimensionless) *quantum* variables corresponding to (x, p) are (ξ, π), where $\xi = \sqrt{m\omega/\hbar}\, x, \pi = bp/\hbar = -id/d\xi$ (see the earlier lecture on the simple harmonic oscillator).

Now, the annihilation operator a is defined by $a = (\xi + i\pi)/\sqrt{2}$, so the classical circular motion in the(x,p) plane evidently corresponds to the expectation value of a rotating at frequency ω in the complex plane. In fact, this is precisely the behavior of the operator a in the Heisenberg representation! Therefore, to find a state approximating the classical behavior, we need to examine eigenstates of a.

Some Exponential Operator Algebra

As a preliminary task, we shall establish some operator identities that prove useful both in understanding the eigenstates of a and in later work.

Suppose that the commutator of two operators A, B

$$[A,B] = c,$$

where c commutes with A and B, usually it's just a number, for instance 1 or $i\hbar$.
Then

$$\begin{aligned}\left[A,e^{\lambda B}\right] &= \left[A,1+\lambda B+(\lambda^2/2!)B^2+(\lambda^3/3!)B^3+\ldots\right]\\ &= \lambda c+(\lambda^2/2!)2Bc+(\lambda^3/3!)3B^2c+\ldots\\ &= \lambda c e^{\lambda B}.\end{aligned}$$

That is to say, the commutator of A with $e^{\lambda B}$ is proportional to $e^{\lambda B}$ itself.

That is reminiscent of the simple harmonic oscillator commutation relation $[H,a^\dagger]=\hbar\omega a^\dagger$ which led directly to the ladder of eigenvalues of H separated by $\hbar\omega$. Will there be a similar "ladder" of eigenstates of A in general?

Assuming A (which is a general operator) has an eigenstate $|a>$ with eigenvalue a,

$$A|a>=a|a>.$$

Applying $[A,e^{\lambda B}]=\lambda c e^{\lambda B}$ to the eigenstate $|a>$:

$$Ae^{\lambda B}|a>=e^{\lambda B}A|a>+\lambda c e^{\lambda B}|a>=(a+\lambda c)e^{\lambda B}|a>.$$

Therefore, unless it is identically zero, $e^{\lambda B}|a>$ is *also* an eigenstate of A, with eigenvalue $a+\lambda c$. We conclude that instead of a *ladder* of eigenstates, we can apparently generate a whole *continuum* of eigenstates, since λ can be set arbitrarily! We shall soon return to this puzzling result, with an example.

To find more operator identities, premultiply $[A,e^{\lambda B}]=\lambda c e^{\lambda B}$ by $e^{-\lambda B}$ to find:

$$\begin{aligned} e^{-\lambda B}Ae^{\lambda B} &= A+\lambda[A,B] \\ &= A+\lambda c. \end{aligned}$$

This identity is *only* true for operators A, B whose commutator c is a number. (Well, c *could* be an operator, provided it still commutes with both A and B).

Our next task is to establish the following very handy identity, which is also only true if $[A,B]$ commutes with A and B:

$$e^{A+B}=e^A e^B e^{-\frac{1}{2}[A,B]}.$$

The proof (due to Glauber, given in Messiah) is as follows:

Take $f(x)=e^{Ax}e^{Bx}$,

$$\begin{aligned} df/dx &= Ae^{Ax}e^{Bx}+e^{Ax}e^{Bx}B \\ &= f(x)\left(e^{-Bx}Ae^{Bx}+B\right) \\ &= f(x)\left(A+x[A,B]+B\right). \end{aligned}$$

It is easy to check that the solution to this first-order differential equation equal to one at $x=0$ is

$$f(x) = e^{x(A+B)} e^{\frac{1}{2}x^2[A,B]}$$

so taking $x = 1$ gives the required identity, $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$.

It also follows that $e^B e^A = e^A e^B e^{-[A,B]}$ provided—as always—that $[A,B]$ commutes with A and B.

Eigenstates of the Annihilation Operator

Recall that the annihilation operator a applied to a simple harmonic oscillator energy eigenstate moves down one step of the ladder, and since the energy eigenvalues cannot be negative, $\hat{a}$ annihilates the lowest state,

$$\hat{a}\,|\,0> = 0.$$

It is worth recalling the wave function representation of this result, with
$\hat{\xi} = \sqrt{m\omega/\hbar}\,\hat{x},\ \hat{\pi} = b\hat{p}/\hbar = -id/d\xi,\ \hat{a} = \left(\hat{\xi} + i\hat{\pi}\right)/\sqrt{2}$,

$$\hat{a}\,|\,0> = \frac{1}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left(\xi + \frac{d}{d\xi}\right) f(\xi) = 0$$

and the solution is $f(\xi) = e^{-\xi^2/2}$, suitably normalized.
What about eigenstates of a with nonzero eigenvalues? Let us write

$$\frac{1}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left(\xi + \frac{d}{d\xi}\right) f(\xi) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \lambda f(\xi)$$

so we've absorbed the normalization factor into the eigenvalue, and we can now cancel it from both sides, to give

$$\frac{df(\xi)}{d\xi} = \left(\sqrt{2}\lambda - \xi\right) f(\xi)$$

easily integrated to give $f(\xi) = Ce^{-\left(\sqrt{2}\lambda - \xi\right)^2/2}$

(Strictly speaking, I should not equate the *vector* $\hat{a}|0\rangle$ with a *function* of ξ, which is *not* a vector—I should have written $\langle x|\hat{a}|0\rangle$ on the left, and it should be understood that the function of ξ is to be translated into a function of x.)

The eigenstates of the annihilation operator

$$\hat{a}\,|\,\lambda >= \lambda\,|\,\lambda >$$

with nonzero eigenvalues have *the same wave function as the ground state, but centered somewhere else*—at $\xi = \sqrt{2}\lambda$, in fact, where λ is the eigenvalue of a. But note that these eigenstates are *not* delta functions, like the eigenstates of the position operator—they are, like the ground state of the oscillator, states with the least possible uncertainty, $\Delta p \cdot \Delta x = \frac{1}{2}\hbar$, as compact as possible in the (x,p) phase space. These states are also not orthogonal to each other, in fact any two of them have nonzero overlap. That does not contradict earlier results on Hermitian operators, because a isn't Hermitian.

There is one big difference between these states $|\lambda\rangle$ and the ground state $|0\rangle$: the ground state is also an eigenstate of the Hamiltonian, so the only time dependence is in the overall phase factor. That is emphatically *not* the case for states with nonzero λ: they correspond to the pendulum having been pulled to one side, and it will begin to swing. This is the motion we are going to analyze using the quantum formalism.

The Translation Operator

We have established that if $A|a> = a|a>$ and $[A,B] = c$, a number, then $e^{\lambda B}|a>$ is an eigenstate of A with eigenvalue $a + \lambda c$. How does this relate to the continuum of eigenstates of the annihilation operator we have just found? In other words, if we take A to be the annihilation operator $\hat{a}$, is there an operator corresponding to B so that $e^{\lambda B}$ translates the center of the wave function a distance λ in x-space? The answer is yes, and it is easily found from the Taylor series:

$$f(\xi+\eta) = f(\xi) + \eta\frac{d}{d\xi}f(\xi) + \frac{\eta^2}{2!}\frac{d^2}{d\xi^2}f(\xi) + \ldots$$

In an obvious notation:

$$f(\xi+\eta) = e^{\eta\frac{d}{d\xi}}f(\xi)$$

so the appropriate operator B is just $d/d\xi$.

Now we've discovered the operator that shifts a wave function along the x-axis by a prescribed amount, the so-called translation operator, we can use it to transform the state $|0\rangle$ to the state $|\lambda\rangle$:

$$|\lambda> = e^{-\sqrt{2}\lambda\frac{d}{d\xi}}|0>.$$

Now, from the introductory section above (or the previous lecture),

$$\frac{d}{d\xi} = i\pi = \frac{a-a^\dagger}{\sqrt{2}}$$

so the translation operator can be written in terms of the annihilation and creation operators:

$$|\lambda> = e^{-\lambda(a-a^\dagger)}|0> = e^{\lambda a^\dagger}e^{-\lambda a}e^{-\lambda^2/2}|0>$$

where in the second step we have used $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$. Since $a|0> = 0$, $e^{-\lambda a}|0> = 1$, so

$$|\lambda> = e^{-\lambda^2/2} e^{\lambda a^\dagger} |0>.$$

From this, we can express $|\lambda\rangle$ as a sum over the eigenstates of the Hamiltonian. Expanding the exponential,

$$|\lambda> = e^{-\lambda^2/2}\left(1 + \lambda a^\dagger + \frac{\left(\lambda a^\dagger\right)^2}{2!} + \dots\right)|0>$$

and recalling that the normalized energy eigenstates are

$$|n> = \frac{\left(a^\dagger\right)^n}{\sqrt{n!}}|0>$$

we find

$$|\lambda> = e^{-\lambda^2/2}\left(|0> + \lambda|1> + \frac{\lambda^2}{\sqrt{2!}}|2> + \frac{\lambda^3}{\sqrt{3!}}|3> + \dots\right).$$

It is easy to check that this state is correctly normalized.

Time Development of an Eigenstate of *a*

Now that we have expressed the eigenstate $|\lambda>$ as a sum over the eigenstates $|n>$ of the Hamiltonian, finding its time development is straightforward: since $|n(t)> = e^{-ni\omega t}|n>$,

$$|\lambda(t)> = e^{-\lambda^2/2}\left(|0> + \lambda e^{-i\omega t}|1> + \frac{\lambda^2 e^{-2i\omega t}}{\sqrt{2!}}|2> + \frac{\lambda^3 e^{-3i\omega t}}{\sqrt{3!}}|3> + \dots\right),$$

which can be written

$$|\lambda(t)> = e^{-\lambda^2/2} e^{\lambda e^{-i\omega t} a^\dagger}|0>.$$

Having established the time dependence, we can now go back to ξ-space, using $a^\dagger = \left(\xi - i\pi\right)/\sqrt{2} = \left(\xi - d/d\xi\right)/\sqrt{2}$ to find

$$|\lambda\left(t\right)> = e^{-\lambda^2/2} e^{\lambda e^{-i\omega t}\left(\xi - d/d\xi\right)/\sqrt{2}}|0> = e^{-\lambda^2/2} e^{\lambda\xi e^{-i\omega t}/\sqrt{2}} e^{-\left(\lambda e^{-i\omega t}/\sqrt{2}\right)d/d\xi} e^{-\lambda^2 e^{-2i\omega t}/4}|0>$$

where we have used $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$ in the second step.

The Taylor series, and hence the shift operator, will still be valid for a shift by a complex amount, so the wavefunction is:

$$\psi_\lambda(\xi,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\lambda^2/2} e^{\lambda\xi e^{-i\omega t}/\sqrt{2}} e^{-\left(\xi-\lambda e^{-i\omega t}/\sqrt{2}\right)^2/2} e^{-\lambda^2 e^{-2i\omega t}/4}.$$