Physics 751 Homework #2

Due Friday September 12, 2008 11:00 am.

1. (a) Defining the delta function as the limit of a narrow Gaussian wave packet (see the web notes on Fourier Series, etc.) prove it has the following properties:

$$\int \delta(x)dx = 1, \quad \delta(x) = 0 \text{ for } x \neq 0.$$

$$\delta(x) = \delta(-x), \quad \delta(ax) = \frac{1}{|a|}\delta(x),$$

$$\int \delta(a-x)\delta(x-b)dx = \delta(a-b).$$

(b) Suppose you define the delta function by:

$$\Delta(x) = \lim_{L\to 0}\Delta_L(x), \text{ where } \Delta_L(x) = 0 \text{ for } |x| \geq L/2, \;\; \Delta_L(x) = 1/L \text{ for } |x| < L/2.$$

Does this function have all the above properties?

2. Use *Mathematica*, *Maple* or Integral Tables to find the integral of (sin$x$)/x from 0 to π and from 0 to infinity. Use your result to estimate the overshoot that appears in a Fourier series representation of a step function (Gibbs' phenomenon).

3. Suppose at t = 0, a free particle of mass m, in one dimension, has a Gaussian wavefunction

$$\psi(x,t=0) = \frac{1}{\left(\pi\Delta^2\right)^{1/4}} e^{-x^2/2\Delta^2}.$$

By taking a Fourier transform and putting in the explicit time-dependence for each plane wave component, find the form of the wavefunction as a function of time, and provide a physical interpretation in terms of finding the particle somewhere.

4. Denoting the lowest energy eigenstate in an infinite square well by ψ_0, and the first excited state by ψ_1, describe the behavior of the probability distribution as a function of time for the state $\psi_0 + \psi_1$ (appropriately normalized).

Find the probability current at the midpoint of the well as a function of time.

How would your analysis be different for the state $\psi_0 + i\psi_1$?

5. Prove **Parseval's Theorem**:

$$\text{If } f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} a(k) e^{ikx}, \text{ then } \int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} \frac{dk}{2\pi} |a(k)|^2.$$

6. Prove the rule for the Fourier Transform of a **convolution** of two functions:

$$\text{If } f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} a(k) e^{ikx}, \quad g(x) = \int_{-\infty}^{\infty} \frac{dk'}{2\pi} b(k') e^{ik'x},$$

$$\text{then } \int_{-\infty}^{\infty} f(x-x') g(x') dx' = \int_{-\infty}^{\infty} \frac{dk}{2\pi} a(k) b(k) e^{ikx}.$$