Physics 751 Homework #9

1. \[ |z\rangle = e^{-\frac{\hat{H}^2}{2}} \left( |0\rangle + z|1\rangle + \frac{z^2}{\sqrt{2!}} |2\rangle + \frac{z^3}{\sqrt{3!}} |3\rangle + \ldots \right) . \]

**Exercise:** Check that this state is correctly normalized, and is an eigenstate of \( \hat{a} \).

2. Prove using an algebraic identity that \( e^{(z\hat{a}^\dagger - z'\hat{a})}|0\rangle \) is an eigenstate of \( \hat{a} \). Is it also an eigenstate of \( \hat{a}^\dagger \)? Prove your assertion.

2. Prove that if \[ |z\rangle = e^{-\frac{\hat{H}^2}{2}} \left( |0\rangle + z|1\rangle + \frac{z^2}{\sqrt{2!}} |2\rangle + \frac{z^3}{\sqrt{3!}} |3\rangle + \ldots \right) , \]
the unit operator \( I = \iint \frac{dxdy}{\pi} |z\rangle \langle z| \)

3. Prove that \( e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \) is correct up to terms \( A^3 \) and \( B^3 \) by expanding the exponentials on both sides and comparing.

4. How does a (position) translation operator affect a wave function expressed in momentum space, \( \psi(p) \)? What is the operator that shifts the momentum space wave function \( \psi(p) \) to \( \psi(p - p_0) \)? How does that operator change \( \psi(x) \)?

5. Prove:
\[ f(x) = e^{x\hat{a}^\dagger} \hat{B} e^{-x\hat{a}} = \hat{B} + x [A, \hat{B}] + \frac{x^2}{2!} [A, [A, \hat{B}]] + \ldots \]
by writing the Taylor series for \( f(x) \) and finding the successive derivatives at the origin.

A unitary squeeze operator is defined by:
\[ U(\theta) = e^{\theta/2(\hat{a} + x \hat{a}^\dagger)} . \]

Use the result for \( f(x) \) above to prove that:
\[ U^\dagger(\theta) \hat{a} U(\theta) = \hat{a} \cosh \theta - \hat{a}^\dagger \sinh \theta, \]
\[ U^\dagger(\theta) \hat{a}^\dagger U(\theta) = -\hat{a} \sinh \theta + \hat{a}^\dagger \cosh \theta. \]

Deduce that
\[ U^\dagger(\theta) \hat{x} U(\theta) = e^{-\theta} \hat{x}, \quad U^\dagger(\theta) \hat{p} U(\theta) = e^\theta \hat{p}, \]

so for positive \( \theta \), the wave function is scaled down—squeezed—in \( x \) space, but simultaneously expanded in \( p \) space, as it must be, since it was a minimum uncertainty packet.

Is it still a minimum uncertainty packet? Is it still an eigenstate of the annihilation operator? If not, what is it an eigenstate of? How do you think it develops in time?