

Div, Grad and Curl in Orthogonal Curvilinear Coordinates

The treatment here is standard, following that in Abraham and Becker, *Classical Theory of Electricity and Magnetism*.

Problems with a particular symmetry, such as cylindrical or spherical, are best attacked using coordinate systems that take full advantage of that symmetry. For example, the Schrödinger equation for the hydrogen atom is best solved using spherical polar coordinates. For this and other differential equation problems, then, we need to find the expressions for differential operators in terms of the appropriate coordinates.

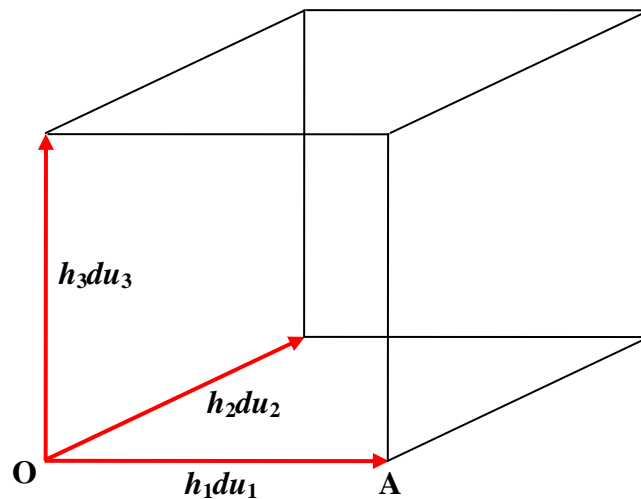
We only look at orthogonal coordinate systems, so that locally the three axes (such as r, θ, φ) are a mutually perpendicular set.

We denote the curvilinear coordinates by (u_1, u_2, u_3) . The standard Cartesian coordinates for the same space are as usual (x, y, z) .

Suppose now we take an infinitesimally small cube with edges parallel to the local curvilinear coordinate directions, and therefore with faces satisfying $u_i = \text{constant}$, $i = 1, 2, 3$ for the three pairs of faces.

The lengths of the edges are then $h_1 du_1$, $h_2 du_2$ and $h_3 du_3$, where h_1, h_2, h_3 are in general functions of u_1, u_2, u_3 . That is to say, the distance across the cube from one corner to the opposite corner

$$ds^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2 = dx^2 + dy^2 + dz^2$$



It is clear that the gradient of a function ψ in the u_1 direction is

$$(\nabla \psi)_1 = \lim_{du_1 \rightarrow 0} \frac{\psi(A) - \psi(0)}{h_1 du_1} = \frac{1}{h_1} \frac{\partial \psi}{\partial u_1}.$$

The divergence of a vector field \mathbf{V} in curvilinear coordinates is found using Gauss' theorem, that the total vector flux through the six sides of the cube equals the divergence multiplied by the volume of the cube, in the limit of a small cube.

The area of the face bracketed by $h_2 du_2$ and $h_3 du_3$ is $h_2 du_2 h_3 du_3$. For that face, the component of the vector field contributing to the flow from the cube is $-V_1$, so the flow across the face is $-V_1 h_2 h_3 du_2 du_3$. To find the flow across the opposite (parallel) face of the cube, corresponding to an increase in u_1 of du_1 , we must bear in mind that h_2 , h_3 and V_1 all vary with u_1 , so the flow will be:

$$V_1 h_2 h_3 du_2 du_3 + \frac{\partial}{\partial u_1} (h_2 h_3 V_1) du_1 du_2 du_3$$

The first term here of course cancels the contribution from the other face. The remaining term, plus the terms with 123 replaced with 231 and 312 from the two other pairs of opposite faces, must, applying Gauss' theorem, add to give $\nabla \cdot \mathbf{V} \times \text{volume} = \nabla \cdot (h_1 h_2 h_3 du_1 du_2 du_3)$.

This gives:

$$\nabla \cdot \mathbf{V} = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1} (h_2 h_3 V_1) + \frac{\partial}{\partial u_2} (h_3 h_1 V_2) + \frac{\partial}{\partial u_3} (h_1 h_2 V_3) \right\}.$$

Putting this together with the expression for the gradient gives immediately the expression for the Laplacian operator in curvilinear coordinates:

$$\nabla^2 \psi = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \psi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial \psi}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \psi}{\partial u_3} \right) \right\}$$

The curl of a vector field \mathbf{A} is found by integrating around one of the square faces. Thus, the 1-component of $\text{curl} \mathbf{A}$ is given by integrating $\mathbf{A} \cdot d\mathbf{s}$ around the (23) square with two of its sides $h_2 du_2$ and $h_3 du_3$. The integral must equal $(\text{curl} \mathbf{A})_1$ multiplied by the area $h_2 du_2 h_3 du_3$. This gives

$$(\text{curl} \mathbf{A})_1 = \frac{1}{h_2 h_3} \left\{ \frac{\partial}{\partial u_2} (A_3 h_3) - \frac{\partial}{\partial u_3} (A_2 h_2) \right\}$$

Cylindrical Coordinates:

$$x = r \cos \varphi$$

$$y = r \sin \varphi$$

$$z = z$$

$$ds^2 = dr^2 + r^2 d\varphi^2 + dz^2$$

Here $(u_1, u_2, u_3) = (r, \varphi, z)$, and $(h_1, h_2, h_3) = (1, r, 1)$.

Therefore, for example,

$$\nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \varphi^2} + \frac{\partial^2 \psi}{\partial z^2}$$

Spherical Polar Coordinates:

$$x = r \sin \theta \cos \varphi$$

$$y = r \sin \theta \sin \varphi$$

$$z = r \cos \theta$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2$$

So $(u_1, u_2, u_3) = (r, \theta, \varphi)$ and $(h_1, h_2, h_3) = (1, r, r \sin \theta)$.

Here

$$\nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \varphi^2}$$