Finding the Prefactor in the Simple Harmonic Oscillator Propagator

Michael Fowler 10/24/07

Hand-waving Argument

Recall that the free particle propagator has the form

$$U\left(x,T;x',0\right)=\sqrt{\frac{m}{2\pi\hbar iT}}\exp\left\{\frac{im\left(x-x'\right)^2}{2\hbar T}\right\}.$$

From a classical mechanical evaluation of the action, we can show the simple harmonic oscillator propagator has the form

$$U\left(x,T;x',0\right)=A\left(T\right)\exp\left\{\frac{im\omega}{2\hbar\sin\omega T}\left[\left(x^2+x'^2\right)\cos\omega T-2xx'\right]\right\}$$

where $A(T)$ is a so far unknown function of time. For sufficiently small T, though, much less than the period of the oscillator, the potential will have no significant effect, so the propagator must tend to the free particle propagator. That is, for T tending to zero, $A(T)=\sqrt{m/2\pi\hbar iT}$. But we also know that at time $t=2\pi/\omega$, all simple harmonic wavefunctions return to their $t = 0$ values, so a particle localized at x' will be localized at x' again after one period. In the exponent we do indeed see this cyclic behavior. In the prefactor, we must replace T by $\left(\sin\omega t\right)/\omega$, that is,

$$A(T)=\sqrt{\frac{m\omega}{2\pi\hbar i\sin\omega T}}.$$

Note this expression is also consistent with the free particle propagator in the limit ω going to zero, in other words, the vanishing of the simple harmonic oscillator potential.

Approximating Integrals by Stationary Phase Techniques

Actually, this result can be derived from the integral over the fluctuations about the classical path. The argument is closely analogous to that for the free particle, and the following equation is a straightforward generalization of that case (discussed in the previous lecture):

$$\langle x|U(T,0)|x'\rangle = \int D[y(t)]e^{iS[x_{cl}(t)+y(t)]/\hbar},$$

$$S[x_{cl}(t)+y(t)] = \int_0^T \tfrac{1}{2}m\left[\left(\dot{x}_{cl}(t)+\dot{y}(t)\right)^2 - \omega^2\left(x_{cl}(t)+y(t)\right)^2\right]dt$$

$$= S[x_{cl}(t)] + \int_0^T m\left(\dot{x}_{cl}(t)\dot{y}(t) - \omega^2 x_{cl}(t)y(t)\right)dt + \int_0^T \tfrac{1}{2}m\left(\dot{y}^2(t) - \omega^2 y^2(t)\right)dt.$$

Just as for the free particle, the middle term in that last line is zero, as it has to be since it's the linear term in the path deviation from the stationary point. As before, the *y*-path (the deviation from the classical least action path) is zero at the two endpoints, so we can integrate by parts to get

$$\int_0^T m\left(\dot{x}_{cl}(t)\dot{y}(t) - \omega^2 x_{cl}(t)y(t)\right)dt = \int_0^T m\left(-\ddot{x}_{cl}(t) - \omega^2 x_{cl}(t)\right)y(t)\,dt$$

identically zero from the classical equation of motion (after all, this is how it's *derived* from least action!)

The hard part is evaluating the integral over paths in the third term. To see how to do this, it's worth briefly reviewing a stationary phase integral in ordinary space, that is,

$$I = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \ldots \int_{-\infty}^{\infty} dx_N \, \exp\{i\, f(x_1,\ldots,x_N)\}$$

where we assume that the real function f has an absolute minimum at the point $(x_1^0, \ldots, x_N^0)$ and take variables $y_i = x_i - x_i^0$. Then to leading order near the stationary point, $f(x) = f(x^0) + \tfrac{1}{2} y_i A_{ij} y_j$. Taking only this term, the integral becomes

$$I = e^{if(x^0)} \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dy_2 \ldots \int_{-\infty}^{\infty} dy_N \, \exp\{\tfrac{1}{2} i y_i A_{ij} y_j\}.$$

The matrix A is positive definite and symmetric, it has real orthogonal eigenvectors, and *we can choose our coordinate axes in the directions of those eigenvectors*. In this case, A is a diagonal matrix, and its diagonal elements are just its eigenvalues λ_i.

So I becomes:

$$I = e^{if(x^0)} \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dy_2 \ldots \int_{-\infty}^{\infty} dy_N \, \exp\{\tfrac{1}{2} i \lambda_i y_i^2\} = e^{if(x^0)} \prod_1^N \sqrt{\frac{2\pi}{i\lambda_i}} = e^{if(x^0)} \left(\frac{2\pi}{i}\right)^{N/2} (\mathrm{Det}\, A)^{-1/2}.$$

Recall that Det A is invariant under an orthogonal transformation, and so is unaffected by transforming back from the eigenvector basis to the original coordinates.

Applying Stationary Phase to the Integral Over Paths

We're now ready to generalize this result from an integral over an N-dimensional space of y_i's to an integral over the infinite dimensional space of paths $y(t)$ with boundary conditions $y(0) = 0$ and $y(t) = 0$. Note that for the particular case at hand, the Simple Harmonic Oscillator, the leading order quadratic term is in fact the whole story, and so gives the *exact* result. This is not, of course, true for general quantum systems.

As a preliminary step, we integrate by parts to write

$$\int_0^T \left(\frac{dy}{dt}\right)^2 dt = -\int_0^T y(t)\frac{d^2}{dt^2}y(t).$$

The path integral then becomes:

$$A(T) = \int_0^0 \exp\left[\frac{i}{\hbar}\int_0^T y(t)\left[\tfrac{1}{2}m\left(-\frac{d^2}{dt^2}-\omega^2\right)\right]y(t)dt\right]D[y(t)].$$

Comparing this with the finite-dimensional version, we see that the N component vector $\{y_i\}$ is replaced by the continuous function $y(t)$, and the matrix A by a differential operator acting on the space of path functions $y(t)$ equal to zero at $t = 0$ and $t = T$.

The operator is well-defined and Hermitian, with eigenstates and eigenvalues:

$$\left(-\frac{d^2}{dt^2}-\omega^2\right)\sin\left(\frac{n\pi t}{T}\right) = \left(\frac{n^2\pi^2}{T^2}-\omega^2\right)\sin\left(\frac{n\pi t}{T}\right).$$

Any reasonable path satisfying the boundary conditions can be written as a sum over these eigenstates with Fourier coefficients y_n,

$$y(t) = \sum_{n=1}^{\infty} y_n \frac{1}{\sqrt{2}}\sin\left(\frac{n\pi t}{T}\right).$$

The functional integral over all paths now becomes an infinite product of ordinary integrals over the variables y_n. This is precisely analogous to the finite integral above, except that there is no maximum value N. The operator is diagonal with respect to this base formed of its eigenstates, and therefore can be represented as an infinite diagonal matrix with the diagonal elements equal to the eigenvalues—and, just as for the finite matrix case, the gaussian integrals can be carried out in succession, each one giving a term

$$\int_{-\infty}^{\infty} dy_n \exp\frac{im}{2\hbar}\lambda_n y_n^2 = \sqrt{\frac{2\hbar\pi}{im\lambda_n}}\,, \text{ where } \lambda_n = \left(\frac{n^2\pi^2}{T^2} - \omega^2\right).$$

When we take the infinite product of all these terms, we find in the denominator the square root of the product of all the eigenvalues—and that product is the determinant of the operator. That is to say,

$$\left[\mathrm{Det}\left(-\frac{d^2}{dt^2} - \omega^2\right)\right]^{-1/2} = \prod_{n=1}^{\infty}\left(\frac{n^2\pi^2}{T^2} - \omega^2\right)^{-1/2} = K(T)\prod_{n=1}^{\infty}\left(1 - \frac{\omega^2 T^2}{n^2\pi^2}\right)^{-1/2} = K(T)\left(\frac{\sin\omega T}{\omega T}\right)^{-1/2}.$$

In the last step, we used the infinite-product representation for the sine function:

$$\sin x = x\prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right).$$

The rather disconcerting factor $K(T)$, which does not depend on ω and hence has nothing to do with the dynamics, can be thrown in with similar time-dependent and constant factors involved in the measure of the integral over paths, and their overall contribution can be nailed down by the simple observation that for $\omega = 0$, we must recover the known free-particle propagator. So, we can be quite careless about overall multiplying factors!

Therefore, matching to the free-particle propagator in the limit $\omega = 0$, we find $A(T) = \sqrt{m\omega / 2\pi\hbar i \sin\omega T}$ for the simple harmonic oscillator.

I used an excellent book, Solitons and Instantons, *by R. Rajaraman, for this section. The original work was all by Feynman.*