Charged Particle in a Magnetic Field

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Introduction

Classically, the force on a charged particle in electric and magnetic fields is given by the Lorentz force law:

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v} \times \vec{B}}{c}\right)$$

This velocity-dependent force is quite different from the conservative forces from potentials that we have dealt with so far, and the recipe for going from classical to quantum mechanics—replacing momenta with the appropriate derivative operators—has to be carried out with more care. We begin by demonstrating how the Lorentz force law arises classically in the Lagrangian and Hamiltonian formulations.

Laws of Classical Mechanics

Recall first (or look it up in Shankar, Chapter 2) that the Principle of Least Action leads to the Euler-Lagrange equations for the Lagrangian L:

$$\frac{d}{dt}\left(\frac{\partial L(q_i, \dot{q}_i)}{\partial \dot{q}_i}\right) - \frac{\partial L(q_i, \dot{q}_i)}{\partial q_i} = 0, \quad q_i,\ \dot{q}_i \text{ being coordinates and velocities.}$$

The canonical momentum p_i is defined by the equation

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

and the Hamiltonian is defined by performing a Legendre transformation of the Lagrangian:

$$H(q_i, p_i) = \sum p_i \dot{q}_i - L(q_i, \dot{q}_i)$$

It is straightforward to check that the equations of motion can be written:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

These are known as *Hamilton's Equations*. Note that if the Hamiltonian is independent of a particular coordinate q_i, the corresponding momentum p_i remains constant. (Such a coordinate is termed *cyclic*, because the most common example is an angular coordinate in a spherically symmetric Hamiltonian, where angular momentum remains constant.)

For the conservative forces we have been considering so far, $L = T - V$, $H = T + V$, with T the kinetic energy, V the potential energy.

Poisson Brackets

Any dynamical variable f in the system is some function of the q_i's and p_i's and (assuming it does not depend explicitly on time) its development is given by:

$$\frac{d}{dt}f(q_i, p_i) = \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i = \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} = \{f, H\}.$$

The curly brackets are called *Poisson Brackets*, and are defined for any dynamical variables as:

$$\{A, B\} = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}.$$

We have shown from Hamilton's equations that for any variable $\dot{f} = \{f, H\}$.
It is easy to check that for the coordinates and canonical momenta,

$$\{q_i, q_j\} = 0 = \{p_i, p_j\}, \quad \{q_i, p_j\} = \delta_{ij}.$$

This was the classical mathematical structure that led Dirac to link up classical and quantum mechanics: he realized that the Poisson brackets were the classical version of the commutators, so a classical canonical momentum must correspond to the quantum differential operator in the corresponding coordinate.

Particle in a Magnetic Field

The Lorentz force is velocity dependent, so cannot be just the gradient of some potential. Nevertheless, the classical particle path is still given by the Principle of Least Action. The electric and magnetic fields can be written in terms of a scalar and a vector potential:

$$\vec{B} = \vec{\nabla} \times \vec{A}, \quad \vec{E} = -\vec{\nabla}\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}.$$

The right Lagrangian turns out to be:

$$L = \tfrac{1}{2}m\vec{v}^2 - q\varphi + \frac{q}{c}\vec{v}.\vec{A}.$$

(*Note*: if you're familiar with Relativity, the interaction term here looks less arbitrary: the relativistic version would have the relativistically invariant $(q/c)\int A^\mu dx_\mu$ added to the action integral, where the four-potential $A_\mu = (\vec{A}, \varphi)$ and $dx_\mu = (dx_1, dx_2, dx_3, cdt)$. This is the simplest possible invariant interaction between the electromagnetic field and the particle's four-velocity. Then in the nonrelativistic limit, $(q/c)\int A^\mu dx_\mu$ just becomes $\int q(\vec{v}.\vec{A}/c - \varphi)dt$.)

The derivation of the Lorentz force from this Lagrangian is given by Shankar on page 84. We give the (equivalent) derivation from the Hamilton equations below.

Note that for zero vector potential, the Lagrangian has the usual $T - V$ form.

For this one-particle problem, the general coordinates q_i are just the Cartesian co-ordinates $x_i = \left(x_1, x_2, x_3\right)$, the position of the particle, and the $\dot{q}_i$ are the three components $\dot{x}_i = v_i$ of the particle's velocity.

The important *new* point is that the canonical momentum

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{x}_i} = mv_i + \frac{q}{c} A_i$$

is no longer mass $\times$ velocity—there is an extra term!

The Hamiltonian is

$$\begin{aligned} H(q_i, p_i) &= \sum p_i \dot{q}_i - L(q_i, \dot{q}_i) \\ &= \sum \left(mv_i + \frac{q}{c} A_i \right) v_i - \tfrac{1}{2} m\vec{v}^2 + q\varphi - \frac{q}{c} \vec{v}.\vec{A} \\ &= \tfrac{1}{2} m\vec{v}^2 + q\varphi \end{aligned}$$

Reassuringly, the Hamiltonian just has the familiar form of kinetic energy plus potential energy. However, to get Hamilton's equations of motion, the Hamiltonian has to be expressed solely in terms of the coordinates and canonical momenta. That is,

$$H = \frac{\left(\vec{p} - q\vec{A}(\vec{x}, t) / c \right)^2}{2m} + q\varphi(\vec{x}, t)$$

where we have noted explicitly that the potentials mean those at the position $\vec{x}$ of the particle at time t.

Let us now consider Hamilton's equations

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}$$

It is easy to see how the first equation comes out, bearing in mind that

$$p_i = mv_i + \frac{q}{c} A_i = m\dot{x}_i + \frac{q}{c} A_i.$$

The second equation yields the Lorentz force law, but is a little more tricky. The first point to bear in mind is that *dp*/*dt* is *not* the acceleration, the *A* term also varies in time, and in a quite complicated way, since it is *the field at a point moving with the particle*. That is,

$$\dot{p}_i = m\ddot{x}_i + \frac{q}{c}\dot{A}_i = m\ddot{x}_i + \frac{q}{c}\left(\frac{\partial A_i}{\partial t} + v_j \nabla_j A_i\right).$$

The right-hand side of the second Hamilton equation $\dot{p}_i = -\dfrac{\partial H}{\partial x_i}$ is

$$-\frac{\partial H}{\partial x_i} = \frac{\left(\vec{p} - q\vec{A}(\vec{x},t)/c\right)}{m} \cdot \frac{q}{c} \cdot \frac{\partial \vec{A}}{\partial x_i} - q\frac{\partial \varphi(\vec{x},t)}{\partial x_i}$$

$$= \frac{q}{c} v_j \nabla_i A_j - q\nabla_i \varphi.$$

Putting the two sides together, the Hamilton equation reads:

$$m\ddot{x}_i = -\frac{q}{c}\left(\frac{\partial A_i}{\partial t} + v_j \nabla_j A_i\right) + \frac{q}{c} v_j \nabla_i A_j - q\nabla_i \varphi.$$

Using $\vec{v} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{v}.\vec{A}) - (\vec{v}.\vec{\nabla})\vec{A}, \quad \vec{B} = \vec{\nabla} \times \vec{A},$ and the expressions for the electric and magnetic fields in terms of the potentials, the Lorentz force law emerges:

$$m\ddot{\vec{x}} = q\left(\vec{E} + \frac{\vec{v} \times \vec{B}}{c}\right)$$

Quantum Mechanics of a Particle in a Magnetic Field

We make the standard substitution:

$$\vec{p} = -i\hbar\vec{\nabla}, \quad \text{so that } [x_i, p_j] = i\hbar\delta_{ij} \quad \text{as usual: but now } p_i \neq mv_i.$$

This leads to the novel situation that the velocities in different directions *do not commute*. From

$$mv_i = -i\hbar\nabla_i - qA_i/c$$

it is easy to check that

$$[v_x, v_y] = \frac{iq\hbar}{m^2 c} B$$

To actually solve Schrödinger's equation for an electron confined to a plane in a uniform perpendicular magnetic field, it is convenient to use the Landau gauge,

$$\vec{A}(x,y,z) = (-By,0,0)$$

giving a constant field B in the z direction. The equation is

$$H\psi(x,y) = \left[\frac{1}{2m}(p_x + qBy/c)^2 + \frac{p_y^2}{2m}\right]\psi(x,y) = E\psi(x,y).$$

Note that x *does not appear in this Hamiltonian*, so it is a cyclic coordinate, and p_x is conserved. In other words, this H commutes with p_x, so H and p_x have a common set of eigenstates. We know the eigenstates of p_x are just the plane waves $e^{ip_x x/\hbar}$, so the common eigenstates must have the form:

$$\psi(x,y) = e^{ip_x x/\hbar}\chi(y).$$

Operating on this wavefunction with the Hamiltonian, the operator p_x appearing in H simply gives its eigenvalue. That is, the p_x in H just becomes a number! Therefore, writing $p_y = -i\hbar d/dy$, the y-component $\chi(y)$ of the wavefunction satisfies:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dy^2}\chi(y) + \tfrac{1}{2}m\left(\frac{qB}{mc}\right)^2(y-y_0)^2\chi(y) = E\chi(y)$$

Where

$$y_0 = -cp_x/qB.$$

We now see that the conserved canonical momentum p_x in the x-direction is actually the coordinate of the center of a simple harmonic oscillator potential in the y-direction! This simple harmonic oscillator has frequency $\omega = |q|B/mc$, so the allowed values of energy for a particle in a plane in a perpendicular magnetic field are:

$$E = (n+\tfrac{1}{2})\hbar\omega = (n+\tfrac{1}{2})\hbar\,|q|\,B/mc.$$

The frequency is of course the cyclotron frequency—that of the classical electron in a circular orbit in the field (given by $mv^2/r = qvB/c, \quad \omega = v/r = qB/mc$) .

Let us confine our attention to states corresponding to the lowest oscillator state, $E = \tfrac{1}{2}\hbar\omega$. How many such states are there? Consider a square of conductor, area $A = L_x \times L_y$, and, for simplicity, take periodic boundary conditions. The center of the oscillator wave function y_0 must lie between 0 and L_y. But remember that $y_0 = -cp_x/qB$, and with periodic boundary conditions $e^{ip_x L_x/\hbar} = 1$, so $p_x = 2n\pi\hbar/L_x = nh/L_x$. This means that y_0 takes a series of evenly-spaced discrete values, separated by

$$\Delta y_0 = ch / qBL_x.$$

So the total number of states $N = L_y / \Delta y_0$,

$$N = \frac{L_x L_y}{\left(\dfrac{hc}{qB}\right)} = A.\frac{B}{\Phi_0},$$

where Φ_0 is called the "flux quantum". So the total number of states in the lowest energy level $E = \frac{1}{2}\hbar\omega$ (usually referred to as the lowest Landau level, or *LLL*) is exactly equal to the total number of flux quanta making up the field *B* penetrating the area *A*.

It is instructive to find y_0 from a purely classical analysis.

Writing $m\dot{\vec{v}} = \frac{q}{c}\vec{v}\times\vec{B}$ in components,

$$m\ddot{x} = \frac{qB}{c}\dot{y},$$
$$m\ddot{y} = -\frac{qB}{c}\dot{x}.$$

These equations integrate trivially to give:

$$m\dot{x} = \frac{qB}{c}(y - y_0),$$
$$m\dot{y} = -\frac{qB}{c}(x - x_0).$$

Here (x_0, y_0) are the coordinates of the center of the classical circular motion (the velocity vector $\dot{\vec{r}} = (\dot{x}, \dot{y})$ is always perpendicular to $(\vec{r} - \vec{r}_0)$) , and $\vec{r}_0$ is given by

$$y_0 = y - cmv_x / qB = -cp_x / qB$$
$$x_0 = x + cmv_y / qB = x + cp_y / qB.$$

(Recall that we are using the gauge $\vec{A}(x, y, z) = (-By, 0, 0)$, and $p_x = \frac{\partial L}{\partial \dot{x}} = mv_x + \frac{q}{c}A_x$, etc.)

Just as y_0 is a conserved quantity, so is x_0: it commutes with the Hamiltonian since

$$\left[x + cp_y / qB, p_x + qBy / c\right] = 0.$$

However, x_0 and y_0 *do not commute with each other*:

$$[x_0, y_0] = -i\hbar c / qB.$$

This is why, when we chose a gauge in which y_0 was sharply defined, x_0 was spread over the sample. If we attempt to localize the point (x0, y0) as well as possible, it is fuzzed out over an area essentially that occupied by one flux quantum. The natural length scale of the problem is therefore the magnetic length defined by

$$l = \sqrt{\frac{\hbar c}{qB}}.$$

References: the classical mechanics at the beginning is similar to Shankar's presentation, the quantum mechanics is closer to that in Landau.