Take-home final exam (Phys 861; Fall 2005)

1. Thermodynamic potential of an ideal Fermi gas can be found from the following formula

$$\Omega = -T \int \ln\left(1 + e^{-\varepsilon(\mathbf{p})/T}\right) \frac{d^d\mathbf{p}}{(2\pi)^d},$$

where d is the dimensionality of the system. Starting from this expression, prove that the specific heat is a linear function of temperature at $T \ll E_{\mathrm{F}}$. Calculate the proportionality constant γ in three and two dimensions, $C = \gamma T$.

2. Consider the following Hamiltonian

$$\hat{\mathcal{H}} = \sum_{\mathbf{p},\sigma} \frac{\mathbf{p}^2}{2m} \hat{c}^\dagger_{\mathbf{p}\sigma} \hat{c}_{\mathbf{p}\sigma} + \frac{1}{2} \sum_{\mathbf{q}} V(\mathbf{q}) \hat{\rho}_{\mathbf{q}} \hat{\rho}_{-\mathbf{q}} + \sum_{\mathbf{q}} V_{\mathrm{ext}}(\mathbf{q}) \hat{\rho}_{\mathbf{q}},$$

where $\hat{c}$ and $\hat{c}^\dagger$ are electron annihilation and creation operators, $\hat{\rho}_{\mathbf{q}} = \sum\limits_{\mathbf{p},\sigma} \hat{c}^\dagger_{\mathbf{p}+\frac{\mathbf{q}}{2}\sigma} \hat{c}_{\mathbf{p}-\frac{\mathbf{q}}{2}\sigma}$ is the electron density operator, $V(q)$ is the two-particle interaction, and $V_{\mathrm{ext}}(q)$ is the external potential.

The current operator is defined as

$$\hat{\mathbf{J}}_{\mathbf{q}} = \sum_{\mathbf{p},\sigma} \frac{\mathbf{p}}{m} \hat{c}^\dagger_{\mathbf{p}+\frac{\mathbf{q}}{2}\sigma} \hat{c}_{\mathbf{p}-\frac{\mathbf{q}}{2}\sigma}$$

Prove the following operator identity (the continuity equation):

$$\frac{\partial \hat{\rho}_{\mathbf{q}}}{\partial t} - i\mathbf{q} \cdot \hat{\mathbf{J}}_{\mathbf{q}} = \hat{0}$$

3. For a one-dimensional harmonic oscillator with mass m and frequency ω, calculate the retarded G^R_{AB}, advanced G^A_{AB}, time-ordered G_{AB}, and temperature (Matsubara) $\mathcal{G}_{AB}$ Green's functions for the following choices of the operators $\hat{A}$ and $\hat{B}$:

   (a) Both $\hat{A}$ and $\hat{B}$ are equal to the position operator $\hat{x}$.

   (b) The operator $\hat{A}$ is equal to the position operator $\hat{x}$ and $\hat{B}$ is equal to the momentum operator $\hat{p}$.

   (c) The operator $\hat{A}$ is equal to the annihilation operator $\hat{a}$ and $\hat{B}$ is equal to the creation operator $\hat{a}^\dagger$.

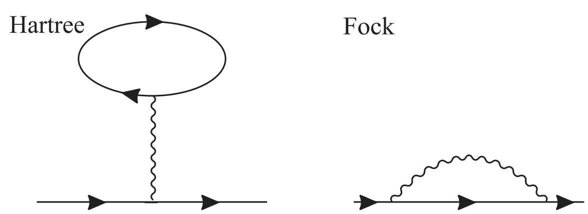
   The definitions of the Green's functions are reminded below

   - Retarded, $G^R_{AB}(t, t') = -i\theta(t - t') \left\langle \left[\hat{A}(t), \hat{B}(t')\right] \right\rangle$

- Advanced, $G^A_{AB}(t,t') = i\theta(t'-t)\left\langle \left[\hat{A}(t), \hat{B}(t')\right]\right\rangle$
- Time-ordered, $G_{AB}(t,t') = -i\left\langle T_t\left(\hat{A}(t)\ \hat{B}(t')\right)\right\rangle$
- Matsubara, $\mathcal{G}_{AB}(\tau,\tau') = -\left\langle T_\tau\left(\hat{A}(\tau)\ \hat{B}(\tau')\right)\right\rangle$

In above definitions $A(t)/A(\tau)$ are Heisenberg operators in real/imaginary time, T_t/T_τ are real/imaginary time-ordering operators, $\langle\ldots\rangle$ indicates the quantum-mechanical averaging, and $[\cdot\ ,\ \cdot]$ is a commutator.

4. Consider a three-dimensional system of fermions, which interact with each other with a point-like potential $V(\mathbf{r}-\mathbf{r}') = u_0\delta(\mathbf{r}-\mathbf{r}')$. Calculate the fermionic self-energy in the Hartree-Fock approximation. I.e., calculate the following diagrams



Within the Hartree-Fock approximation, derive the general formula for the correction to the chemical potential in terms of the spin s, the fermion density n, and the interaction strength u_0. Note that in the model of "spinless fermions," the correction vanishes. Can this fact be understood without calculations?

Hint: Note that the integral of the Green's function over energy is simply the (Fermi) distribution function.

5. Consider a three-dimensional system of fermions interacting with each other via a long range potential $v(r) = g^2/r^2$, where r is the distance between the particles and g is a small constant. Doing the standard RPA perturbation theory find the screened potential in real space and calculate the spectrum of collective modes.

Hint: This problem is very similar to problem 3 of your mid-term (screening of Coulomb interaction in two dimensions).

6. *Difficult problem (extra credit):* The conductivity of a disordered normal metal can be calculated from the classical Boltzmann equation or even from Newton laws (see lectures). The result for the DC conductivity is (Drude formula)

$$\sigma = \frac{ne^2\tau}{m},$$

where n is the density of carriers, e is the charge, m is the effective mass of carriers, and τ is the scattering time.

The same formula can be derived from the linear response Kubo theory; However calculations are more difficult: To find the conductivity one has to calculate the current-current response function. In the presence of an external field described by the vector-potential $\mathbf{A}(\mathbf{r}, t)$, the current operator is

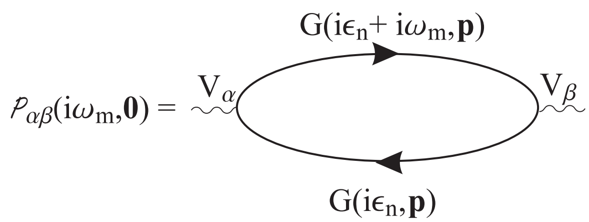
$$\hat{\mathbf{j}}(\mathbf{r}, t) = \frac{e}{2m}\left[-i\hat{\psi}^\dagger(\mathbf{r}, t)\nabla\hat{\psi}(\mathbf{r}, t) + \text{h. c.}\right] - \frac{e^2}{mc}\mathbf{A}(\mathbf{r}, t)\hat{\psi}^\dagger(\mathbf{r}, t)\hat{\psi}(\mathbf{r}, t)$$

We see that in the presence of an external field the current consists of two parts: the usual gradient part and the diamagnetic term.

The electrical conductivity is determined by the current-current correlation function, which has the form (in Matsubara representation)

$$\mathcal{K}^{jj}_{\alpha\beta}(i\omega_n, \mathbf{r} - \mathbf{r}') = \mathcal{P}_{\alpha\beta}(i\omega_n, \mathbf{r} - \mathbf{r}') + \frac{ne^2}{m}\delta_{\alpha\beta}\delta(\mathbf{r} - \mathbf{r}'), \qquad (*)$$

where $\mathcal{P}_{\alpha\beta}$ is the correlator described by the diagram



This is a linear response correlator corresponding to the gradient part of the current. In a disordered metal, the Matsubara Green's function has the following form:

$$\mathcal{G}(\varepsilon_n, \mathbf{p}) = \frac{1}{i\varepsilon_n - \xi_\mathbf{p} + i/(2\tau)\text{sgn}\,\varepsilon_n},$$

where τ is the scattering time due to impurities (let us assume that only point impurities are present). Note that the diagram contains velocities $v_\alpha = p_\alpha/m$ in the vertices.

Using the expression for the disorder-averaged Green's function...

(a) Prove the following identity $\mathcal{P}_{\alpha\beta}(i\omega_n = 0, \mathbf{q} = \mathbf{0}) = -\frac{ne^2}{m}\delta_{\alpha\beta}$. This thus will prove that the zero-frequency bubble exactly cancels the second term in Eq. (*) above.

(b) Since the electric field is related to the vector potential as $\mathbf{E} = -\frac{1}{c}\dot{\mathbf{A}}$, the Matsubara conductivity can be defined as

$$\sigma_{\alpha\beta}(i\omega_n) = \frac{1}{\omega_n}\left[\mathcal{P}_{\alpha\beta}(i\omega_n, \mathbf{0}) - \mathcal{P}_{\alpha\beta}(0, \mathbf{0})\right].$$

Calculate the Matsubara conductivity.

Hint: In solving this part of the problem, use the ξ-approximation, *i.e.* convert the integral over momentum into an integral over $\xi = v_{\rm F}(p - p_{\rm F})$.

(c) To find the conductivity as a function of the physical frequency, one has to do an analytical continuation: I.e., analytically continue the function $\sigma_{\alpha\beta}(i\omega_n)$ from the upper complex plane ($n > 0$) to real frequencies. This can be done simply by replacing $i\omega_n \to \omega$. Using this procedure, calculate the AC conductivity and reproduce the standard Drude formula in the $\omega \to 0$ limit.

Reading: Abrikosov, Gor'kov, and Dzyaloshinskii, Mahan, and Lectures

Due Friday, December 16 (9:00am, room PHS 313)

Each student gives a 20-30 minute presentation on one of the problems.