Problem set 2

1. Consider 3 (three) free bose particles in a box $L \times L \times L$. Assuming periodic boundary conditions, find single-particle eigenstates $\phi_n(\mathbf{r})$. Using these functions, find many-body wave functions and their energies.

2. Consider a classical chain of oscillators (see also your lecture notes)

$$\mathcal{H} = \sum_{i=-\infty}^{\infty} \left[\frac{p_i^2}{2m_i} + \frac{K}{2} (x_i - x_{i+1})^2 \right], \tag{1}$$

where $m_i = m$ if i is even and $m_i = M > m$, if i is odd. Find the speed of sound in the chain, using the Laplace formula

$$c^2 = \frac{\partial P}{\partial \rho},$$

where P is the pressure and ρ is the density. Note that in 1D, pressure and force are the same thing.

3. Consider a quantum chain of oscillators (see also your lecture notes) described by the Hamiltonian (1) above, but with $\hat{x}_i$ and $\hat{p}_i = -i\hbar \frac{d}{dx_i}$ understood as quantum-mechanical operators and $m_i \equiv m = M$. Diagonalize the Hamiltonian by quantizing the classical normal modes.

   Hint: Fourier transform the position operator and the momentum operator and find the corresponding commutation relations. Verify that there is an oscillator corresponding to each Fourier harmonics in the Hamiltonian. Introduce creation and annihilation operators and find the spectrum.

4. Using Bogoliubov transformations, diagonalize the following fermionic Hamiltonian ($J_{1,2}$ and B are some constants):

$$\mathcal{H} = \sum_{i=-\infty}^{\infty} \left(J_1 a_i^\dagger a_{i+1} + J_2 a_i a_{i+1} - B a_i^\dagger a_i + \text{ h. c. } \right).$$

   This Hamiltonian appears in the context of a one-dimensional quantum magnetic (so-called XY-model). Find the spectrum of quasiparticles $\varepsilon(k)$ of this Hamiltonian. Note that for $J_1 = J_2$ and $B = 0$ the dispersion disappears. Can this fact be understood without calculations?

Due Thursday, September 8 (in class)