Problem set 4

1. The displaysment operator of longitudinal waves in a medium (longitudinal phonons) has the form

$$\hat{\mathbf{q}}(\mathbf{r},t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \frac{\mathbf{k}}{|\mathbf{k}|} \left\{ \hat{q}_{\mathbf{k}} e^{i[\mathbf{kr}-\omega_0(\mathbf{k})t]} + \hat{q}^{\dagger}_{\mathbf{k}} e^{-i[\mathbf{kr}-\omega_0(\mathbf{k})t]} \right\}$$

   The corresponding momentum operator is $\rho\dot{\hat{\mathbf{q}}}$. These operators satisfy the following commutation relations: $\left[\hat{q}_\alpha(\mathbf{r},t), \rho\dot{\hat{q}}_\beta(\mathbf{r}',t)\right] = i\delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}')$

   (a) Verify that the operators $\hat{b}_{\mathbf{k}}$ and $\hat{b}^{\dagger}_{\mathbf{k}}$ defined by the relation $\hat{b}_{\mathbf{k}} = \sqrt{2\rho\omega_0(\mathbf{k})}\hat{q}_{\mathbf{k}}$ satisfy the usual bosonic commutation relations.

   (b) Express the kinetic energy of longitudinal oscillations in terms of $\hat{b}$ and $\hat{b}^{\dagger}$

$$\hat{K} = \frac{\rho}{2} \int d^3\mathbf{r} \dot{\hat{\mathbf{q}}}^2$$

2. Fermionic Green's function is defined (as zero temperature) as

$$G_{\alpha\beta}(x;x') = -i \left\langle T \left(\psi_{H\alpha}(x) \psi^{\dagger}_{H\beta}(x') \right) \right\rangle,$$

   where ψ_H and $\psi^{\dagger}_H$ are field operators in the Heisenberg representation, α and β are the spin indices, and $x = (t, \mathbf{r})$.

   (a) Prove that the density of fermions can be related to the Green's function as

$$n(x) = -i \lim_{t' \to t+0} G_{\alpha\alpha}(t,\mathbf{r};t',\mathbf{r}')$$

   (b) Using this formula for the density and the Greeen's function in momentum representation $G(\varepsilon,\mathbf{p})$, find the Fermi-momentum of free spinless fermions with the density n.

3. Electrons in a two-dimensional quantum well are often described by the Rashba model, defined by the following Hamiltonian

$$\mathcal{H}_{\text{Rashba}} = \sum_{\mathbf{p},\sigma,\sigma'} \left(\frac{\mathbf{p}^2}{2m} + \alpha \mathbf{e_z} \left[\boldsymbol{\tau}_{\sigma,\sigma'} \times \mathbf{p} \right] \right) C^{\dagger}_{\mathbf{p}\sigma} C_{\mathbf{p}\sigma'},$$

   where α is a constant (spin-orbit coupling), $\mathbf{e_z}$ is a unit vector in the z-direction, and $\boldsymbol{\tau}$ are the Pauli matrices. Note that $\mathbf{p}$ is a two-dimensional vector. Diagonalize the Hamiltonian and find the Green's function for the non-interacting Rashba electrons.

Reading: Abrikosov, Gor'kov, and Dzyaloshinskii
Due Thursday, September 29 (in class)