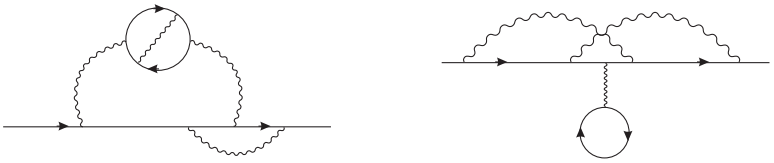
Prooblem set 5

1. Phonon excitations in a crystal are often described by the Debye model in which the phonon spectrum is $\omega(\mathbf{k}) = c|\mathbf{k}|$, if $k < k_{\mathrm{D}}$ (where ω_{D} is the Debye frequency) and $\omega(k) = 0$, if $k > k_{\mathrm{D}}$. Find the threshold Debye frequency in a crystal of volume V, which contains N identical atoms. Assume that only longitudinal phonons are present.

2. Draw all possible topologically non-equivalent diagrams (connected and disconnected) in second order perturbation theory with respect to a two-particle interaction $V(\mathbf{r}_1 - \mathbf{r}_2)$.

3. Write down the analytical expressions (in momentum representation) corresponding to the following diagrams:



4. A localized magnetic impurity is introduced in a Fermi gas. The spin interacts with electrons via the following exchange interaction

$$\mathcal{H}_{\mathrm{int}} = J \int S^i \delta(\mathbf{r}) \psi^\dagger_\alpha(\mathbf{r}) \sigma^i_{\alpha\beta} \psi_\beta(\mathbf{r}) d^3\mathbf{r} \equiv J S^i \hat{\sigma}^i(\mathbf{r} = \mathbf{0}),$$

where J is a constant, which is assumed *small*, $\mathbf{S}$ is the impurity spin, $i = x, y, z$, α and β are spin indices, and $\sigma^i_{\alpha\beta}$ are the Pauli matrices. Find the spin polarization $\left\langle \hat{\sigma}^i(\mathbf{r}) \right\rangle$ at large distances from the spin ($rp_{\mathrm{F}} \gg 1$) in leading order of perturbation theory with respect to the exchange interaction.

Hint: Express the polarization density through the Green's function as follows $\left\langle \hat{\sigma}^i(\mathbf{r}) \right\rangle = \lim\limits_{t' \to t+0} \left[-i \sigma^i_{\alpha\beta} G_{\beta\alpha}(\mathbf{r}, t; \mathbf{r}', t') \right]$. Calculate the first order correction $G_{\alpha\beta}(\varepsilon, \mathbf{r}; \mathbf{r}') = J S^i \sigma^i_{\alpha\beta} G_0(\varepsilon, \mathbf{r}) G_0(\varepsilon, -\mathbf{r}')$ in real space.

Reading: Abrikosov, Gor'kov, and Dzyaloshinskii
Due Thursday, October 13 (in class)