Problem set 7

1. Consider a *one-dimensional* (1D) Fermi gas in an external AC field, which couples to density

$$\hat{\mathcal{H}}_{\rm int}(t) = -\int \phi(t,x)\hat{n}(t,x)dx.$$

The linear response of the density $\hat{n}(t,x) = \hat{\psi}^\dagger(t,x)\hat{\psi}(t,x)$ to the external field $\phi(x,t)$ is defined by

$$\langle \hat{n}(t,x)\rangle = \int dt'dx' Q(t-t',x-x')\phi(t',x')$$

or $\langle \hat{n}(\omega,q)\rangle = Q(\omega,q)\phi(\omega,q)$ in Fourier space. Using the Kubo formula, calculate the generalized susceptibility (density-density response). Find the susceptibility $Q(\omega,q)$ for small q and ω: $q \ll p_{\rm F}$ and $\omega \ll E_{\rm F}$.

2. The imaginary part of the spin susceptibility of a Fermi gas has the form (see your lecture notes)

$$\operatorname{Im}\chi(\omega,\mathbf{q}) = \pi\mu_{\rm B}^2\nu_0\frac{\omega}{v_{\rm F}q}\theta\left(qv_{\rm F} - |\omega|\right).$$

Using Kramers-Krönig relations, find the real part of the susceptibility. Compare the result with your lecture notes.

3. A localized magnetic impurity is introduced in a Fermi gas *at a finite temperature T*. The spin interacts with electrons via the following exchange interaction

$$\mathcal{H}_{\rm int} = J\int S^i\delta(\mathbf{r})\psi^\dagger_\alpha(\mathbf{r})\sigma^i_{\alpha\beta}\psi_\beta(\mathbf{r})d^3\mathbf{r} \equiv JS^i\hat{\sigma}^i(\mathbf{r}=\mathbf{0}),$$

Using the Matsubara Green's function technique, find the suppression of the RKKY (Ruderman-Kittel-Kasuya-Yosida) oscillations by temperature (see also your homework set 5 and the lecture notes). *I.e.*, find the electron spin density $\left\langle \hat{\sigma}^i(\mathbf{r};T)\right\rangle$ as a function of distance r ($p_{\rm F}r \gg 1$) and temperature T.

Hint: Express the polarization density through the Matsubara Green's function as follows $\left\langle \hat{\sigma}^i(\mathbf{r})\right\rangle = \lim_{\tau'\to tau+0}\left[-i\sigma^i_{\alpha\beta}\mathcal{G}_{\beta\alpha}(\mathbf{r},\tau;\mathbf{r}',\tau')\right]$. Calculate the first order correction $\mathcal{G}_{\alpha\beta}(\varepsilon,\mathbf{r};\mathbf{r}') = JS^i\sigma^i_{\alpha\beta}\mathcal{G}_0(\varepsilon_n,\mathbf{r})\mathcal{G}_0(\varepsilon_n,-\mathbf{r}')$ in real space, where

$$\mathcal{G}_0(\varepsilon_n,-\mathbf{p}) = \frac{1}{\varepsilon_n - \xi_{\mathbf{p}}}$$

is the non-interacting Matsubara Green's function and $\varepsilon_n = (2n+1)\pi T$, $n = 0, \pm1, \pm2, \ldots$.

Reading: Abrikosov, Gor'kov, and Dzyaloshinskii and lecture notes
Due Thursday, November 17 (in class)