Physics 862 – Spring 02
Problem set 1 - Due Tuesday, Jan 29

1.

Some drills with second quantization.

a) Let $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be the state with no particle and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ the state with one particle, the particle being an electron of a given spin or the equivalent. Then the creation operator $\hat{c}^\dagger$ is represented by the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$: What matrices represent the following operators:

$$\hat{c};\ \hat{c}\hat{c};\ \hat{c}^\dagger\hat{c}^\dagger;\ \hat{c}^\dagger\hat{c};\ \hat{c}\hat{c}^\dagger;\ \hat{c}^\dagger\hat{c}+\hat{c}\hat{c}^\dagger;\ \hat{c}^\dagger\hat{c}-\hat{c}\hat{c}^\dagger \tag{1}$$

b) By convention, $\hat{c}^\dagger_\uparrow|0\rangle \equiv |\uparrow\rangle$ represents the one-particle state with spin in the z direction. Construct a second-quantized operator, which we can call $\hat{S}_z$, with the following properties: applied to $|0\rangle$ it gives 0, applied to $|\uparrow\rangle$ it gives $\frac{1}{2}|\uparrow\rangle$ and applied to $|\downarrow\rangle$ it gives $-\frac{1}{2}|\downarrow\rangle$. Using only the anticommutation properties of the $\hat{c}_\sigma$ and $\hat{c}^\dagger_\sigma$ operators, show that $\hat{S}_z$ also gives correctly the z component of total spin in the two-particle state $|\uparrow\downarrow\rangle = \hat{c}^\dagger_\uparrow\hat{c}^\dagger_\downarrow|0\rangle$.

c) Construct the second-quantized operators $\hat{S}_x$ and $\hat{S}_y$ and their eigenstates. You may do this any way you like. One possibility is to construct first the raising and lowering operators $\hat{S}_+$ and $\hat{S}_-$ and then use $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$.

2.

Obtain the second-quantized expression for the 3 Cartesian components of the magnetization $\hat{M}(r)$. It should be easy to write down $\hat{M}_z(r)$. For the other 2 components, the results of problem 1 can be helpful.

3.

a) Compute the Jacobian of the transformation $R = (r + r')/2;\ s = r - r'$. It is enough to do this in one dimension (why?).

b) Consider a wire of length L and uniform cross-section A and obtain a second-quantized operator $\hat{I}$ such that $\langle \hat{I} \rangle$ gives the current I flowing in the wire. If the cross-section is not uniform, the current is still constant along the wire, as long as there is no charge accumulation.

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Solution

$$\hat{S}_z = \frac{1}{2}\left(\hat{c}^{\dagger}_{\uparrow}\hat{c}_{\uparrow} - \hat{c}^{\dagger}_{\downarrow}\hat{c}_{\downarrow}\right) \tag{2}$$

$$\hat{S}_z^2 = \frac{1}{4}\left(\hat{c}^{\dagger}_{\uparrow}\hat{c}_{\uparrow} - \hat{c}^{\dagger}_{\downarrow}\hat{c}_{\downarrow}\right)\left(\hat{c}^{\dagger}_{\uparrow}\hat{c}_{\uparrow} - \hat{c}^{\dagger}_{\downarrow}\hat{c}_{\downarrow}\right) = \frac{1}{4} \tag{3}$$