Ch. 1 - Part 1

Lecture 1: Basic Electrostatics

1.1 Coulomb's law, electrostatic field

The force between two charges is given by Coulomb's law

$$F = \kappa \frac{q_1 q_2}{r^2}$$

where κ depends on the choice of units. In cgs electrostatic units, $\kappa = 1$, F is in dynes, r in centimeters, q_1 and q_2 in statcoulombs. In the MKS system, F is in newtons, r in meters, q_1 and q_2 in coulombs, and

$$\kappa[\mathrm{MKS}] = \frac{1}{4\pi\epsilon_0} = 8.988 \times 10^9 \,\mathrm{nt} \cdot \mathrm{m}^2/\mathrm{coul}^2$$

Thus two coulombs at the distance of one meter repel each other with the very large force of 9×10^9 nt or 9×10^{14} dy (we will usually write 9 instead of 8.988... in the following).

Two coulombs at one cm repel each other with the force of 9×10^{18} dy. Hence

$$1\,\mathrm{coul} = 2.998 \times 10^9 \,\mathrm{statcoul}$$

If charge q is at point $\mathbf{x}$ and charge q_1 is at point $\mathbf{x}_1$, the Coulomb force is directed along the vector $\mathbf{x} - \mathbf{x}_1$:

$$\mathbf{F}_{on\,q} = q q_1 \frac{\mathbf{x} - \mathbf{x}_1}{|\mathbf{x} - \mathbf{x}_1|^3} \tag{1.1}$$

This force can be regarded as the product of q times the electric field $\mathbf{E}$ produced by q_1 at $\mathbf{x}$. $\mathbf{E}$ depends on $\mathbf{r} = \mathbf{x} - \mathbf{x}_1$ and we can write

$$\mathbf{E}(\mathbf{r}) = q_1 \frac{\mathbf{r}}{r^3} = -q_1 \nabla \frac{1}{r}$$

For an arbitrary distribution of charges q_i , we have

$$\mathbf{E}(\mathbf{x}) = -\sum_i q_i \nabla \frac{1}{|\mathbf{x} - \mathbf{x}_i|}$$

and for a continuous charge distribution, $\rho(\mathbf{x}') = \sum_i q_i \delta(\mathbf{x}' - \mathbf{x}_i)$:

$$\mathbf{E}(\mathbf{x}) = -\int \rho(\mathbf{x}') \nabla \frac{1}{|\mathbf{x} - \mathbf{x}'|} d^3x'$$

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Insert on properties of δ-function as needed

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1.2 Gauss's law

If the surface S encloses the charges q_i

$$\oint_S \mathbf{E} \cdot \mathbf{n} \; da = 4\pi \sum_i q_i$$

where the normal $\mathbf{n}$ is directed outwards. This is proved by noting that for a single charge q (see Jackson's Fig. 1.2)

$$\mathbf{E} \cdot \mathbf{n} = q \frac{\cos\theta}{r^2}$$

so that the flux of the electric field through the area element da is

$$\mathbf{E} \cdot \mathbf{n} \, da = q \frac{da \; \cos\theta}{r^2} = q d\Omega$$

where $d\Omega$ is the solid angle subtended by da.

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For a continuous distribution

$$\oint_S \mathbf{E} \cdot \mathbf{n} \; da = 4\pi \int_V \rho(\mathbf{x}) d^3x$$

where V is the volume enclosed by S. Using the divergence theorem

$$\int_V \nabla \cdot \mathbf{E} \; d^3x = \oint_S \mathbf{E} \cdot \mathbf{n} \; da$$

we obtain the differential form of Gauss's law

$$\nabla \cdot \mathbf{E} = 4\pi\rho$$

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1.2.1 Applications of Gauss's law

These are left by Jackson as Problems, and can be found in more elementary books. There are three basic types:

- Problems with spherical symmetry, which are solved by noting that on a sphere of radius r the field is radial and given in magnitude by

$$4\pi r^2 E = 4\pi Q$$

  where q is the total charge inside the sphere. To the outside world, it is as if all the charge were in the center of the sphere. The analogous result for the gravitational field is due to Newton.

- Problems with cylindrical symmetry, which similarly give

$$2\pi \rho E = 4\pi \lambda$$

  where λ is the total charge per unit length inside the cylinder of radius ρ.

- "Pillbox" arguments (see Jackson's Fig. 1.4). The best known of these leads to the conclusion that the field at the surface of a conductor (which is normal to the surface) is related to the surface charge density σ by $E = 4\pi\sigma$.

1.3 The potential and Poisson's equation

The field of a unit point charge is $\mathbf{E} = -\nabla(1/r)$. The field of a set of charges can also be written as a gradient:

$$\mathbf{E}(\mathbf{x}) = -\sum_i q_i \nabla \frac{1}{|\mathbf{x} - \mathbf{x}_i|} = -\nabla\Phi(\mathbf{x})$$

where Φ is the electrostatic potential:

$$\Phi(\mathbf{x}) = \sum_i q_i \frac{1}{|\mathbf{x} - \mathbf{x}_i|} = \int \rho(\mathbf{x}') \frac{1}{|\mathbf{x} - \mathbf{x}'|} d^3x'$$

For any function Φ, $\nabla\times(\nabla\Phi) = 0$. Hence

$$\nabla \times \mathbf{E} = 0,$$

The physical meaning of $\Phi(\mathbf{x})$ is as follows: if there are no charges at infinity, so that $\Phi(\infty) = 0$, then $\Phi(\mathbf{x})$ is the work required to bring a unit charge from ∞ to $\mathbf{x}$ (the other charges being fixed). More generally, the work required to bring a charge q from point A to point B is

$$W = -\int_A^B q\mathbf{E}\cdot d\mathbf{l} = q\left[\Phi(B) - \Phi(A)\right]$$

The work done depends only on the end points, not on the path; hence the net work in going around a closed path is zero; the electric field is *conservative.* These properties are just a consequence of $\nabla \times \mathbf{E} = 0$,as can also be seen from *Stokes' theorem:*

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = \int_S \nabla \times \mathbf{E} \cdot \mathbf{n}\, da = 0$$

where S is any surface bounded by the closed contour C.

The esu unit of potential is the statvolt = statcoul/cm = erg/statcoul.

The MKSA unit is the volt = J/coul = 10^7erg/($3 \cdot 10^9$statcoul). Thus

$$\text{volt} = \frac{\text{statvolt}}{300}$$

Starting from Coulomb's law we have derived the two *differential field equations of electrostatics*

$$\begin{aligned} \nabla \times \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{E} &= 4\pi\rho \end{aligned}$$

The most general solution of the first equation can be written $\mathbf{E} = -\nabla\Phi$, where at this point Φ is unspecified. Inserting in the second equation, we find that Φ must satisfy the *Poisson equation:*

$$\nabla^2\Phi = -4\pi\rho \tag{1.2}$$

Conversely, one can show directly that the only solution of the Poisson equation that vanishes at infinity is the familiar

$$\Phi(\mathbf{x}) = \int \rho(\mathbf{x}')\frac{1}{|\mathbf{x} - \mathbf{x}'|}d^3x' \tag{1.3}$$

To prove that (1.3) is a solution of (1.2), we make use of

$$\nabla^2 \frac{1}{|\mathbf{x} - \mathbf{x}'|} = -4\pi\delta(\mathbf{x} - \mathbf{x}'),$$

which is really nothing but Poisson's equation when ρ is a point charge. If this equation is granted, then

$$\nabla^2 \Phi(\mathbf{x}) = \int \rho(\mathbf{x}')\nabla^2 \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|}\right) d^3x' = -4\pi \int \rho(\mathbf{x}')\delta(\mathbf{x} - \mathbf{x}')d^3x'$$

gives immediately $\nabla^2\Phi(\mathbf{x}) = -4\pi\rho(\mathbf{x})$.

Then, putting $\mathbf{r} = \mathbf{x} - \mathbf{x}'$, all we need to show is that

$$\nabla^2 \frac{1}{r} = -4\pi\delta(\mathbf{r})$$

The easiest way to proceed is to note that for $r \neq 0$

$$\nabla^2 \frac{1}{r} = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\frac{1}{r} = 0$$

but for $r = 0$ we have a singularity with integrated strength

$$\int \nabla^2 \frac{1}{r} d^3x = \int_S \left(\nabla \frac{1}{r}\right) \cdot \mathbf{n}\, da$$

by the divergence theorem. The integral is conveniently evaluated on a sphere of finite radius r, and it gives $4\pi R^2\left(-1/R^2\right) = -4\pi$.

Another way to handle the singular behavior is to replace *1/r* with a "regularized" function, such as

$$\frac{1}{\sqrt{r^2 + a^2}}$$

and then let $a \to 0$. This is done in Jackson and may appeal to the rigor-minded. "Regularization" procedures of this type are a way of life in quantum electrodynamics, so Jackson takes the opportunity to show one here.

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1.4 Fields of point, line, and surface charges

- Point charge $q\delta(\mathbf{r}) = q\delta(x)\delta(y)\delta(z)$

$$\mathbf{E} = q\frac{\mathbf{r}}{r^3}$$
$$\Phi = \frac{q}{r}$$

- Straight line charge $\lambda\delta(\rho) = \lambda\delta(x)\delta(y)$

$$\mathbf{E} = 2\lambda\,\frac{\rho}{\rho^2}$$

$$\Phi = -2\lambda\ln\rho = -\lambda\ln\rho^2$$

- Plane charge $\sigma\delta(x)$

$$E_x = 2\pi\sigma\,\mathrm{sign}(x)$$
$$\Phi = -2\pi\sigma\,|x|$$

  That is, the potential has a cusp and the field is discontinuous. For a non-planar charge layer, there is the same discontinuity, but Φ is not symmetric for $\pm x$.

- Plane dipole layer $-D\delta'(x)$

$$E_x = -4\pi D\delta(x)$$
$$\Phi = 2\pi D\,\mathrm{sign}(x)$$

  That is, the potential is discontinuous and the field vanishes, except inside the dipole layer, where it cannot be seen. For a non-planar dipole layer, there is the same discontinuity, but Φ is not antisymmetric for $\pm x$. A simple result applies for any dipole layer of constant D (see Jackson's Fig. 1.7): $-\Phi/D$ is equal to the solid angle subtended by the layer.