2. Ch. 1, part 2

2.1. Electrostatic energy

The work required to bring the charge q_i from infinity to the point $\mathbf{x}_i$ in the presence of existing charges $q_1, \ldots, q_{i-1}$ is

$$W_i = q_i \sum_{j=1}^{i-1} \frac{q_j}{|\mathbf{x}_i - \mathbf{x}_j|}$$

The total work required to assemble N charges is then:

$$W_N = \sum_{i=2}^{N} q_i \sum_{j=1}^{i-1} \frac{q_j}{|\mathbf{x}_i - \mathbf{x}_j|}$$

Each pairwise interaction appears once in this double sum. If we sum over i and j independently, but exclude $i = j$, each pair is counted twice. Thus we can also write

$$W_N = \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{|\mathbf{x}_i - \mathbf{x}_j|} \tag{2.1}$$

Going to a continuum distribution, we have no way to exclude the interaction of an infinitesimal charge element with itself. So we leave it in and write:

$$W = \frac{1}{2} \iint \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, d^3x \, d^3x' \tag{2.2}$$

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For a smooth three dimensional charge distribution, the interaction of a charge element with the nearby charges is

$$\frac{1}{2}\rho(\mathbf{x}) \, d^3x \int_{\mathcal{V}} \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, d^3x' \approx \mathcal{V}^{2/3} \rho^2(\mathbf{x}) \, d^3x$$

and vanishes for $\mathcal{V} \to 0$. So the self-interaction does not contribute and the formula (2.2) is correct, unless $\rho(\mathbf{x})$ contains a point charge, or even a line charge. At the macroscopic level the charge distributions always have a finite extent, really, so there is no problem. At the fundamental level, there is a problem if the elementary particles (leptons, quarks) are truly point charges. Relativistic QED (quantum electrodynamics) deals with this difficulty by postulating a simple negative term that cancels the infinite self-energy in W. Few people believe that this renormalization procedure is the final word.

2.1.1. The field energy

One reason why we like eq. (2.2) is that it can be transformed to

$$W = \frac{1}{8\pi} \int |\boldsymbol{\nabla}\Phi|^2 \, d^3x = \frac{1}{8\pi} \int |\mathbf{E}|^2 \, d^3x \tag{2.3}$$

which shows that the work done in assembling the charges is stored as field energy. To get this expression we rewrite eq. (2.2) as

$$W = \frac{1}{2} \int \rho(\mathbf{x}) \Phi(\mathbf{x}) \, d^3 x,$$

substitute $\rho(\mathbf{x}) = -(1/4\pi)\nabla^2\Phi(\mathbf{x})$, and eliminate $\Phi\nabla^2\Phi(\mathbf{x})$ by using $\boldsymbol{\nabla} \cdot (\Phi\boldsymbol{\nabla}\Phi) = \Phi\nabla^2\Phi + |\boldsymbol{\nabla}\Phi|^2$. What we actually get is

$$W = \frac{1}{8\pi} \int |\boldsymbol{\nabla}\Phi|^2 \, d^3 x - \frac{1}{8\pi} \int_{\mathcal{S}} \Phi \frac{\partial\Phi}{\partial n} \, da$$

So it seems that eq. (2.3) is obtained only if the integral is over all space and there are no charges at infinity, or in special cases such as grounded conductors ($\Phi = 0$) on the boundary $\mathcal{S}$. In practice, however, one can always find an $\mathcal{S}$ such that the boundary term vanishes or is unimportant.

Eq. (2.3) is a fundamental result. It is very useful to think that there really is an energy density

$$w = \frac{|\mathbf{E}|^2}{8\pi}$$

that resides in the field and is greater where the field is stronger. Note that the field energy is always positive. A consequence of this is that

An assembly of charges cannot be in equilibrium under electrostatic forces alone.

We already noted that for point charges (2.1) is not the same as (2.2) or (2.3). For two unlike charges, (2.1) is negative, while (2.3) is positive (and infinite). Jackson discusses this point in greater detail.

2.2. Boundary values

A number of important relations can be obtained by fiddling around with the field energy expression and related integrals. They are very useful to discuss the properties of the $\mathbf{E}$ field in a confined geometry. Typically, the boundaries are made of conductors, each held at a known potential, or carrying a known total charge. The problem is to find the charge density σ on the surface of each conductor and the field $\mathbf{E}$ in the space between them.

2.2.1. Green s first identity

$$\int_{\mathcal{V}} (\phi\nabla^2\psi + \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\psi) \, d^3 x = \oint_{\mathcal{S}} \phi \frac{\partial\psi}{\partial n} \, da \tag{2.4}$$

This is obtained simply by integrating the identity

$$\boldsymbol{\nabla} \cdot (\phi\boldsymbol{\nabla}\psi) = \phi\nabla^2\psi + \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\psi$$

and using the divergence theorem. It does not seem too interesting in itself, but it has many uses. An important corollary is the following.

Uniqueness theorem

The solution of Poisson s equation is unique if Φ is given on the boundary (Dirichlet bound-ary condition). It is unique up to an additive constant if $\partial\Phi/\partial n$ is given on the boundary (Neumann boundary condition). It is also unique if the boundary conditions are Dirichlet on part of the boundary, Neumann on the rest.

   This is proven by assuming that there are two distinct solutions, Φ_1 and Φ_2, and showing that $U = \Phi_1 - \Phi_2$ must vanish. Just use 2.4 with $\phi = \psi = U$

$$\int_{\mathcal{V}} (U\nabla^2 U + \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} U)\, d^3x = \oint_{\mathcal{S}} U\frac{\partial U}{\partial n}\, da$$

and note that $\nabla^2 U = 0$ in all $\mathcal{V}$ and that either $U = 0$ (Dirichlet) or $\partial U/\partial n = 0$ (Neumann) on $\mathcal{S}$. It follows that $|\boldsymbol{\nabla} U|$ vanishes and U is constant in $\mathcal{V}$. The constant must be zero in the Dirichlet case, or for mixed boundary conditions, but is arbitrary in the pure Neumann case.

   By a modification of this argument one can prove that, in charge-free space, the quantity $\int |\boldsymbol{\nabla}\Psi|^2\, d^3x$ is minimal, subject to the appropriate boundary conditions, when Ψ coincides with the potential Φ, solution of *Laplace s equation* $\nabla^2\Phi = 0$. See Jackson s problem 1.14.

2.2.2. Green s theorem and integral equations

Write down Green s first identity for ϕ, ψ and ψ, ϕ:

$$\int_{\mathcal{V}} (\phi\nabla^2\psi - \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\psi)\, d^3x = \oint_{\mathcal{S}} \phi\frac{\partial \psi}{\partial n}\, da$$
$$\int_{\mathcal{V}} (\psi\nabla^2\phi - \boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}\phi)\, d^3x = \oint_{\mathcal{S}} \psi\frac{\partial \phi}{\partial n}\, da$$

and subtract side by side. The result is *Green s second identity*, or Green s theorem:

$$\int_{\mathcal{V}} (\phi\nabla^2\psi - \psi\nabla^2\phi)\, d^3x = \oint_{\mathcal{S}} \left(\phi\frac{\partial \psi}{\partial n} - \psi\frac{\partial \phi}{\partial n}\right)\, da \tag{2.5}$$

This gives a very useful formula if we choose

$$\psi = \frac{1}{R} = \frac{1}{|\mathbf{x} - \mathbf{x}'|}$$

so that $\nabla^2\psi = -4\pi\delta(\mathbf{x} - \mathbf{x}')$, and we identify ϕ with the potential Φ. If $\mathbf{x}$ is inside $\mathcal{V}$ we get

$$-4\pi\Phi(\mathbf{x}') - \int_{\mathcal{V}} \frac{1}{R}\nabla^2\Phi\, d^3x = \oint_{\mathcal{S}} \left(\Phi\frac{\partial}{\partial n}\frac{1}{R} - \frac{1}{R}\frac{\partial \Phi}{\partial n}\right)\, da$$

Recalling that $\nabla^2\Phi = -4\pi\rho$ and interchanging the names of $\mathbf{x}$ and $\mathbf{x}'$:

$$\Phi(\mathbf{x}) = \int_\mathcal{V} \frac{\rho(\mathbf{x}')}{R} d^3x' + \frac{1}{4\pi} \oint_\mathcal{S} \left(\frac{1}{R}\frac{\partial\Phi}{\partial n'} - \Phi(\mathbf{x}')\frac{\partial}{\partial n'}\frac{1}{R} \right) da' \tag{2.6}$$

There is a lot of mathematical physics in this equation:

- If $\mathcal{V}$ is all of space and there are no charges at infinity, it simply gives back Coulomb s law

$$\Phi(\mathbf{x}) = \int_{\mathcal{V}_\infty} \frac{\rho(\mathbf{x}')}{R} d^3x'$$

- If $\partial\Phi/\partial n$ is given on the boundary $\mathcal{S}$ (Neumann), we obtain an integral equation for Φ on $\mathcal{S}$. This equation may be easier to solve than the original Poisson equation. Once $\Phi(\mathbf{x})$ is known for $\mathbf{x}$ on $\mathcal{S}$, it can be found everywhere by integration (in principle, if not in practice).

- If Φ is given on the boundary $\mathcal{S}$ (Dirichlet), we can similarly get an integral equation for $\partial\Phi/\partial n$ on $\mathcal{S}$ and then find $\Phi(\mathbf{x})$ everywhere. The integral equation is

$$\left.\frac{\partial\Phi}{\partial n}\right|_\mathcal{S} = \int_\mathcal{V} \rho(\mathbf{x}')\frac{\partial}{\partial n}\frac{1}{R} d^3x' +$$
$$\frac{1}{4\pi} \oint_\mathcal{S} \left[\left(\frac{\partial}{\partial n}\frac{1}{R}\right)\frac{\partial\Phi}{\partial n'} - \Phi(\mathbf{x}')\frac{\partial^2}{\partial n\partial n'}\frac{1}{R} \right] da'$$

- These integral equations show that it is not possible to assign both Φ and $\partial\Phi/\partial n$ arbitrarily on the boundary. Physically, $\mathcal{S}$ could be the surface of conductors which are held at fixed potentials. The charges in these conductors shift about and determine the surface charge density σ, which is given by $4\pi\sigma = \partial\Phi/\partial n$. Note that $\mathbf{n}$ is directed *into* the conductor (contrary to the usual practice).

- We can view $(1/4\pi)\partial\Phi/\partial n$ as a surface charge density and $-(1/4\pi)\Phi$ as a dipole layer that cause, respectively, a discontinuity in the field and in the potential, in such a way that Φ jumps to zero in crossing $\mathcal{S}$.
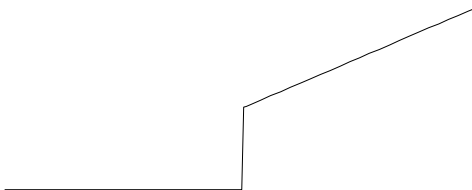
$\mathcal{S}$

2.2.3. Green s function

We can solve the boundary value problem anew for every $\rho(\mathbf{x})$, but there is a better way. Consider for instance the Dirichlet case. We can construct the general solution if we can solve the particular case when $\mathcal{S}$ is at zero potential and $\rho(\mathbf{x})$ is a unit point charge, $\rho(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}')$. This special solution $G(\mathbf{x}, \mathbf{x}')$ is the *Green function*. It obeys the equation

$$\nabla^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}')$$
$$G(\mathbf{x}, \mathbf{x}') = 0 \quad \text{when } \mathbf{x} \text{ is on } \mathcal{S}$$

Once G has been found (this is easier said than done), we can use Green s theorem with $\psi = G$ and $\phi = \Phi$ and proceed as in the derivation of eq.(2.6). In other words, simply write G in place of $1/R$ in (2.6). What we find is an explicit expression for $\Phi(\mathbf{x})$:

$$\Phi(\mathbf{x}) = \int_{\mathcal{V}} G(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')\, d^3x' + \frac{1}{4\pi}\oint_{\mathcal{S}} \Phi(\mathbf{x}')\frac{\partial G}{\partial n'}\, da' \qquad (2.7)$$

In particular, if $\Phi = 0$ on $\mathcal{S}$, we get

$$\Phi(\mathbf{x}) = \int_{\mathcal{V}} G(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')\, d^3x'$$

which is very much like Coulomb s law. In fact, $1/|\mathbf{x} - \mathbf{x}'|$ is nothing but the Green function for all space.

Similarly, one can define and use a Green function $G_N(\mathbf{x}, \mathbf{x}')$ for Neumann boundary conditions. Jackson discusses this case in detail.

If we put $G = 1/R + F$ and we use $\nabla^2 G = -4\pi\delta(\mathbf{x} - \mathbf{x}')$ and $\nabla^2(1/R) = -4\pi\delta(\mathbf{x} - \mathbf{x}')$, we see that F satisfies Laplace s equation $\nabla^2 F = 0$ everywhere in $\mathcal{V}$. We can regard F as an electric potential due to (fictitious) charges located outside $\mathcal{V}$. The idea is to continue F and G across $\mathcal{S}$ as if they were not discontinuous at $\mathcal{S}$. In some cases this approach leads to a simple solution of the boundary value problem (method of images, Chapter 2).

There are many applications of Green s theorems. A useful one is the *mean value theorem,* Jackson s Problem 1.10 (ASSIGNED):

In charge-free space, $\Phi(\mathbf{x})$ is equal to the average of Φ on any sphere centered at $\mathbf{x}$.

A consequence of this is that $\Phi(\mathbf{x})$ cannot have maxima or minima in charge-free space. One could say that this is obvious, because in charge-free space Φ satisfies *Laplace s equation,* $\nabla^2\Phi = 0$, which in cartesian coordinates is

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

so that it is impossible for all the second derivatives to have the same sign. However, the mean value theorem shows that it is also impossible to have a point where all the second and third derivatives vanish but all the fourth derivatives are positive, for instance.