1 Fields in a cavity

Jackson works out in detail the fields in a waveguide, then uses the waveguide solutions to work out the fields in a cavity. His procedure is to break up the cavity fields into the e^{ikz} part and the e^{-ikz} part, work on each separately, and recombine them. This can be cumbersome.

One can obtain formulas that work directly for fields proportional to any combination of sinusoidal or exponential functions. The result shown here is in cartesian coordinates, and thus it is especially suited for box-shaped cavities.

Start from Maxwell's equations written out explicitly:

$$\begin{aligned}
\partial_y H_z - \partial_z H_y &= -i\,(\omega/c)\,\epsilon E_x && (1)\\
\partial_z H_x - \partial_x H_z &= -i\,(\omega/c)\,\epsilon E_y && (2)\\
\partial_x H_y - \partial_y H_x &= -i\,(\omega/c)\,\epsilon E_z && (3)
\end{aligned}$$

$$\begin{aligned}
\partial_y E_z - \partial_z E_y &= i\,(\omega/c)\,\mu H_x && (4)\\
\partial_z E_x - \partial_x E_z &= i\,(\omega/c)\,\mu H_y && (5)\\
\partial_x E_y - \partial_y E_x &= i\,(\omega/c)\,\mu H_z && (6)
\end{aligned}$$

Assuming only that the each field component satisfies $\partial_z^2\phi = -\gamma^2\phi$, we can obtain all the other field components from the "superpotentials" E_z and H_z :

$$\begin{aligned}
\left(\epsilon\mu\,(\omega/c)^2 - \gamma^2\right) E_x &= \partial_x\partial_z E_z + i(\omega/c)\mu\partial_y H_z && (7)\\
\left(\epsilon\mu\,(\omega/c)^2 - \gamma^2\right) E_y &= \partial_y\partial_z E_z - i(\omega/c)\mu\partial_x H_z && (8)\\
\left(\epsilon\mu\,(\omega/c)^2 - \gamma^2\right) H_x &= \partial_x\partial_z H_z - i(\omega/c)\epsilon\partial_y E_z && (9)\\
\left(\epsilon\mu\,(\omega/c)^2 - \gamma^2\right) H_y &= \partial_y\partial_z H_z + i(\omega/c)\epsilon\partial_x E_z && (10)
\end{aligned}$$

To derive (7), for example, all we need is Eq. (1) and the z derivative of (5):

$$\begin{aligned}
\partial_y H_z - \partial_z H_y &= -i\,(\omega/c)\,\epsilon E_x\\
\partial_z^2 E_x - \partial_z\partial_x E_z &= i\,(\omega/c)\,\mu\partial_z H_y
\end{aligned}$$

Eliminate $\partial_z H_y$ from these two equations and replace $\partial_z^2 E_x$ with $-\gamma^2 E_x$.

2 Boundary conditions on E and H

Jackson is rather vague on this point. For a cavity with perfectly conducting walls and end walls perpendicular to the z axis:

$$E_z = 0 \text{ on the side walls and } \partial_z E_z = 0 \text{ on the end walls}$$
$$\partial_n H_z = 0 \text{ on the side walls and } H_z = 0 \text{ on the end walls}$$

To derive these boundary conditions, we note first that $E_{||}$ must be inversely proportional to the conductivity, otherwise the current would be infinite. This implies that on the side walls $E_z = 0$, and on the end walls $E_x = 0$, $E_y = 0$ and their lateral derivatives also vanish. Then we can use $\partial_x E_x = 0$ and $\partial_y E_y = 0$ in the divergence equation $\partial_x E_x + \partial_y E_y + \partial_z E_z = 0$ to conclude that $\partial_z E_z = 0$ on the end walls. We can also use $\partial_x E_y = 0$ and $\partial_y E_x = 0$ in the equation $i\,(\omega/c)\,\mu H_z = \partial_x E_y - \partial_y E_x$, from $-(1/c)\partial\mathbf{B}/\partial t = \nabla \times \mathbf{E}$, to conclude that $H_z = 0$ on the end walls. In general the normal component H_n vanishes on a wall, and so do its lateral derivatives.

Finally, we show that the normal derivative $\partial_n H_z$ vanishes on the side walls. Consider a wall element and orient the x axis normal to it. Use $-i\,(\omega/c)\,\epsilon E_y = \partial_x H_z - \partial_z H_x$, from $(1/c)\partial\mathbf{D}/\partial t = \nabla \times \mathbf{H}$, and note that $E_y = 0$, because it is a parallel component of $\mathbf{E}$, and that $\partial_z H_x = 0$, because it is a lateral derivative of the normal component of $\mathbf{H}$. It follows that $\partial_x H_z = 0$. In general, $\partial_n \mathbf{H}_{||} = 0$ on a wall.