Boost and rotation

We want to write a generic Lorentz matrix Λ as the product RB, where R is a rotation and B a boost. The generic boost matrix is (Jackson, eq. (11.98)):

$$B^{\alpha}_{\ \beta} = \begin{pmatrix} \gamma & -\gamma v^j \\ -\gamma v^i & \delta^i_{\ j} + \gamma^2 v^i v^j/(\gamma+1) \end{pmatrix}$$

where v^i are the Cartesian components of the boost velocity $\mathbf{v}$. On the other hand, R is of the form

$$R^{\alpha}_{\ \beta} = \begin{pmatrix} 1 & 0 \\ 0 & R^i_{\ j} \end{pmatrix}$$

where $R^i_{\ j}$ is a three-dimensional rotation matrix. The problem is to determine B and R, given Λ.

We can keep the indices consistently placed (up or down) by rewriting B in the form

$$B^{\beta}_{\ \gamma} = \begin{pmatrix} \gamma & \gamma v_k \\ -\gamma v^j & \delta^j_{\ k} - \gamma^2 v^j v_k/(\gamma+1) \end{pmatrix}$$

where $v_i = -v^i$ are covariant components. We find

$$\begin{pmatrix} \Lambda^0_{\ 0} & \Lambda^0_{\ k} \\ \Lambda^i_{\ 0} & \Lambda^i_{\ k} \end{pmatrix} = R^{\alpha}_{\ \beta} B^{\beta}_{\ \gamma} = \begin{pmatrix} \gamma & \gamma v_k \\ -\gamma R^i_{\ j} v^j & R^i_{\ k} - \gamma^2 R^i_{\ j} v^j v_k/(\gamma+1) \end{pmatrix}$$

hence $\gamma = \Lambda^0_{\ 0}$, $v_k = \Lambda^0_{\ k}/\gamma$, $R^i_{\ j} v^j = -\Lambda^i_{\ 0}/\gamma$, and

$$R^i_{\ k} = \Lambda^i_{\ k} + \gamma^2 R^i_{\ j} v^j v_k/(\gamma+1) = \Lambda^i_{\ k} - \gamma\, \Lambda^i_{\ 0} v_k/(\gamma+1)$$

or explicitly

$$R^i_{\ k} = \Lambda^i_{\ k} - \Lambda^i_{\ 0}\Lambda^0_{\ k}/(\gamma+1)$$