Approximation of End Wire Radius [1]

See figure 15. Assume the solution has the form

$$\phi(x,y) = X(x)Y(y)$$

Substituting this expression into Laplace's equation gives

$$\nabla^2\phi = YX'' + XY'' = 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$\frac{X''}{X} = \frac{Y''}{Y} = -K$$

where K is a positive constant. The boundary conditions are: $\phi = 0$ at $y = 0, y = a$, and $x = \pm\infty$. If $k^2 = K$ then:

$$Y'' = -k^2 Y$$

Therefore,

$$Y = Asinky + Bcosky$$

Using the boundary condition that $\phi = 0$ at $y = a$:

$$Y = Asinky = 0$$

Therefore, since $A \neq 0$, $ka = n\pi$ and $k = \frac{n\pi}{a}$.

Also, $X'' = k^2 X$. Therefore, $X = Ce^{kx} + De^{-kx}$. Using the boundary conditions, $X = Ce^{kx}$ from $-\infty < x < 0$ and $X = De^{-kx}$ from $0 < x < \infty$. Therefore, the answer from $-\infty < x < 0$ is

$$\phi_- = \sum_{i=1}^{\infty} C_n e^{\frac{n\pi x}{a}} sin(\frac{n\pi y}{a})$$

The answer from $0 < x < \infty$ is

$$\phi_+ = \sum_{i=1}^{\infty} A_n e^{\frac{-n\pi x}{a}} sin(\frac{n\pi y}{a})$$

At x=0, the potential must be continuous. Therefore, $C_n = A_n$.

Represent a line charge by a delta function. Thus, the charge distribution is:

$$\sigma(y) = q\delta(y - d)$$

[1] Wolfgang K. Panofsky and Melba Phillips, Classical Electricity and Magnetism (Reading: Addison-Wesley Publishing Company, Inc. 1962) 53-55.

For two parallel plates $E = \frac{\sigma}{\varepsilon_o}$. Therefore, using F=qE=$\frac{-d\phi}{dx}$:

$$\frac{q\delta(y-d)}{\varepsilon_o} = [\frac{d\phi_-}{dx} - \frac{d\phi_+}{dx}]_{x=0}$$

where q is the charge per unit length.

$$\phi'_- = \sum_{i=1}^{\infty} \frac{n\pi}{a} A_n sin(\frac{n\pi y}{a})$$

$$\phi'_+ = \sum_{i=1}^{\infty} \frac{-n\pi}{a} A_n sin(\frac{n\pi y}{a})$$

$$\phi'_- - \phi'_+ = \sum_{i=1}^{\infty} A_n \frac{2n\pi}{a} sin(\frac{n\pi y}{a})$$

$$\frac{q\delta(y-d)}{\varepsilon_o} = \sum_{i=1}^{\infty} A_n \frac{2n\pi}{a} sin(\frac{n\pi y}{a})$$

Multiply by sin($\frac{m\phi y}{a}$) and integrate from 0 to a:

$$\int_0^a \frac{q\delta(y-d)}{\varepsilon_o} sin(\frac{m\pi y}{a}) = \int_0^a \sum_{i=1}^{\infty} A_n \frac{2n\pi}{a} sin(\frac{n\pi y}{a}) sin(\frac{m\pi y}{a})$$

Due to orthogonality all terms equal 0 except when n=m. Therefore,

$$\frac{q}{\varepsilon_o} sin(\frac{n\pi d}{a}) = A_n \frac{2n\pi}{a} \int_0^a sin^2(\frac{n\pi x}{a})$$

$$\int_0^a sin^2(\frac{n\pi x}{a}) = \frac{a}{2}$$

$$\frac{q}{\varepsilon_o} sin(\frac{n\pi d}{a}) = A_n n\pi$$

$$A_n = \frac{q}{n\pi\varepsilon_o} sin(\frac{n\pi d}{a})$$

Therefore, the solution is

$$\phi_- = \frac{q}{\varepsilon_o \pi} \sum_{i=1}^{\infty} \frac{1}{n} sin(\frac{n\pi d}{a}) e^{\frac{n\pi x}{a}} sin(\frac{n\pi y}{a})$$

$$\phi_+ = \frac{q}{\varepsilon_o \pi} \sum_{i=1}^{\infty} \frac{1}{n} sin(\frac{n\pi d}{a}) e^{\frac{-n\pi x}{a}} sin(\frac{n\pi y}{a})$$

Therefore,

$$q = \frac{\phi_+ \varepsilon_o \pi}{\sum_{i=1}^{\infty} \frac{1}{n} sin(\frac{n\pi d}{a}) e^{\frac{-n\pi x}{a}} sin(\frac{n\pi y}{a})}$$

At the coordinates ($\frac{a}{2}$,r), and using the fact that d=3 mm and a=6 mm, the sum becomes

$$\sum_{i=1}^{\infty} \frac{1}{n} sin(\frac{n\pi}{2}) e^{\frac{-n\pi r}{a}} sin(\frac{n\pi}{2})$$

In this sum, all even terms go to 0. Therefore, the sum becomes

$$\sum_{odd} \frac{1}{n} e^{\frac{-n\pi r}{a}}$$

The electric field on a wire is given by

$$E = \frac{q}{2\pi\varepsilon_o r}$$

Therefore,

$$E = \frac{\phi_+}{2 \sum_{odd} \frac{1}{n} e^{\frac{-n\pi r}{a}} r} \quad (1)$$

To get an estimate of how big the wire should have been, the E given by the previous equation should be less than or equal to the electric field at the center wires. Very close to the anode wires the electric field is given by the equation

$$E = \frac{CV_o}{2\pi\varepsilon_o r} \quad (2)$$

where C is the capacitance per unit length given by the equation

$$C = \frac{2\pi\varepsilon_o}{\frac{\pi l}{s} - ln(\frac{2\pi a}{s})}$$

where s is the distance between wires, l is the cathode-anode distance, and a is the anode wire radius. In the UMS wire chambers, s=.8 mm, l=3 mm, and a=6.25 μm. This gives C=3.76 $\frac{pF}{m}$. Plugging this value into equation 2 gives 3.24x10$^7 \frac{V}{C}$. Therefore using equation 1

$$\frac{\phi_+}{2 \sum_{odd} \frac{1}{n} e^{\frac{-n\pi r}{a}} r} \leq 3.24\text{x}10^7 \frac{\text{V}}{\text{C}}$$

Using V=3000 V,

$$\sum_{odd} \frac{1}{n} e^{\frac{-n\pi r}{a}} r \leq 4.63\text{x}10^7$$

Using the first three terms

$$r(e^{-\frac{\pi r}{a}} + \frac{1}{3}e^{-\frac{3\pi r}{a}} + \frac{1}{5}e^{-\frac{5\pi r}{a}} \leq 4.63\text{x}10^7$$

Solving numerically, r$\approx 31\mu m$. From the integral test the error in this solution is

$$\int_4^\infty \frac{1}{2x-1} e^{\frac{-(2x-1)\pi}{.006}} \approx 0$$

Therefore, the radius of the end wires should be increased to about $31\mu m$ in order to stop the sparking.