Unifying the Forces (and Particles!)

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Outline:

1. SM review: forces, particles, Higgs mechanism
2. Quick review of group theory:
   - representations, generators, roots & weights, etc.
3. The basic GUT idea
4. Choices of GUT groups & their relations
5. SM embedding into SU(5)
6. Gauge coupling unification & the GUT scale
7. Breaking the GUT group – hierarchy problem
8. Other issues and experimental signatures:
   - fermion masses
   - proton decay, B–L conservation
   - cosmological implications
   - extensions: SUSY, SO(10) & neutrinos, ...
9. Modern topics:
   - unification with gravity
   - embedding into string theory
   - GUTs in higher dimensions, TeV-scale GUTs
   - orbifold GUTs

1 SM review

Standard Model is <u>the</u> theory governing all fundamental particles and interactions for $\underset{\sim}{\ell \gtrsim 10^{-18}\,m} \iff \underset{\sim}{E \lesssim 10^2\,\text{GeV}}$.

It is a theory of <u>FORCES</u> & the <u>PARTICLES</u> on which they act.

("verbs") ("nouns")

We shall review only the grossest, "architectural" structure of the SM.

FORCES (verbs)

	strong	⊗	weak	⊗	hypercharge
			"electroweak"		
	$SU(3)_c$	⊗	$SU(2)_W$	⊗	$U(1)_Y$
(spin-1 bosons):	8 gluons	⊗	$A^\pm, A^3$	⊗	B
	$\alpha_3 \approx \frac{1}{8.5}$	⊗	$\alpha_2 \approx \frac{1}{29.6}$	⊗	$\alpha_Y \approx \frac{1}{98.3}$

[measured at Z scale ≈ 90 GeV]

PARTICLE CONTENT

(nouns)

$\Rightarrow$ particles are "CHIRAL FERMIONS":

- FERMIONS : Dirac bispinor ψ
- CHIRAL : definite handedness:

$$\psi_L = \tfrac{1}{2}(1-\gamma_5)\psi \qquad \text{left}$$

$$\psi_R = \tfrac{1}{2}(1+\gamma_5)\psi \qquad \text{right}$$

each is only two components.

Particle content of SM consists of three generations of chiral fermions:

LEFT : electroweak doublets :

$$\begin{pmatrix} u \\ d \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} \begin{pmatrix} t \\ b \end{pmatrix} \quad \leftarrow \text{quarks: each comes in three colors } (r, g, b)$$

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix} \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix} \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} \quad \leftarrow \text{leptons: no colors}$$

RIGHT : all components are singlets :

$$\left.\begin{matrix} (u) & (c) & (t) \\ (d) & (s) & (b) \end{matrix}\right\} \leftarrow \text{quarks: each in three colors}$$

not yet discovered! Assume ν massless!

$$\left.\begin{matrix} \cancel{(\nu_e)} & \cancel{(\nu_\mu)} & \cancel{(\nu_\tau)} \\ (e) & (\mu) & (\tau) \end{matrix}\right\} \leftarrow \text{leptons: no colors}$$

Let's adopt a very succinct notation to describe the transformation properties of the particles with respect to the SM gauge symmetries:
[FIRST GENERATION ONLY ; others just repeat...]

$$
\begin{array}{r|l}
 & SU(3)_c \otimes SU(2)_W \otimes U(1)_Y \\
\hline
Q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L : & (3\,,\ 2)_{1/3} \\
L_L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L : & (1\,,\ 2)_{-1} \\
u_R : & (3\,,\ 1)_{4/3} \\
d_R : & (3\,,\ 1)_{-2/3} \\
e_R : & (1\,,\ 1)_{-2} \\
\nu_{eR} : & (1\,,\ 1)_{0} \quad \leftarrow \text{if it exists}
\end{array}
$$

[Note: u_R is still $\mathbf{3}$, not $\bar{\mathbf{3}}$: different handedness of the same up quark!]

But it's somewhat awkward to deal with some fields listed as (L), others (R)

... obscures the relations between different representations

Recall "charge conjugation" operation: (particle $\longleftrightarrow$ antiparticle)

$$\psi^c \equiv i\gamma^2 \psi^*$$

Then we observe

$$(\psi_R)^c = i\gamma^2 \left[\tfrac{1}{2}(1+\gamma_5)\psi \right]^*$$

since $\gamma_5^* = \gamma_5$

$$= \tfrac{i}{2}\gamma^2 (1+\gamma_5)\psi^*$$

since $\{\gamma^\mu, \gamma^5\} = 0$

$$= \tfrac{1}{2}(1-\gamma_5)\left[i\gamma^2 \psi^* \right]$$

$$= (\psi^c)_L$$

$\Rightarrow$ The conjugate of a right-handed component of a fermion

is the left-handed component of the

conjugate fermion!

thus, if $u_R = (\mathbb{3}, \mathbb{1})_{4/3}$

then $(u_R)^c = (\overline{\mathbb{3}}, \mathbb{1})_{-4/3} = (u^c)_L$

We can thus drop all "L" subscripts and write all fields in terms of left-handed components:

$Q: \quad (\mathbb{3}, \mathbb{2})_{+1/3}$

$L: \quad (\mathbb{1}, \mathbb{2})_{-1}$

$u^c: \quad (\overline{\mathbb{3}}, \mathbb{1})_{-4/3}$

$d^c: \quad (\overline{\mathbb{3}}, \mathbb{1})_{+2/3}$

$e^c: \quad (\mathbb{1}, \mathbb{1})_{+2}$

[$\nu^c: \quad (\mathbb{1}, \mathbb{1})_0$ ... if it exists!]

The SM also has an "ADVERB" — The Higgs sector!

$$SU(3)_c \otimes SU(2)_W \otimes U(1)_Y$$

Weak ($A^{\pm}, A^3$) — hyperchage (B)

$$\underbrace{SU(2)_W \otimes U(1)_Y}_{} \xrightarrow{\langle\phi\rangle \text{ Higgs vev}} U(1)_{EM}$$

$U(1)_{EM}$ (γ = photon)

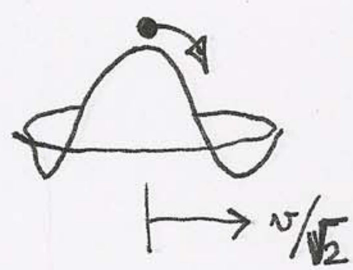
How does this happen?

Higgs field = complex doublet of spin-0 Lorentz scalars:

$$\phi = \begin{pmatrix} \phi^+ \\ \phi_0 \end{pmatrix}_{Y=1} \quad \leftarrow \text{four degrees of freedom}$$

↑ (chosen)

Imagine $V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger \phi)^2$



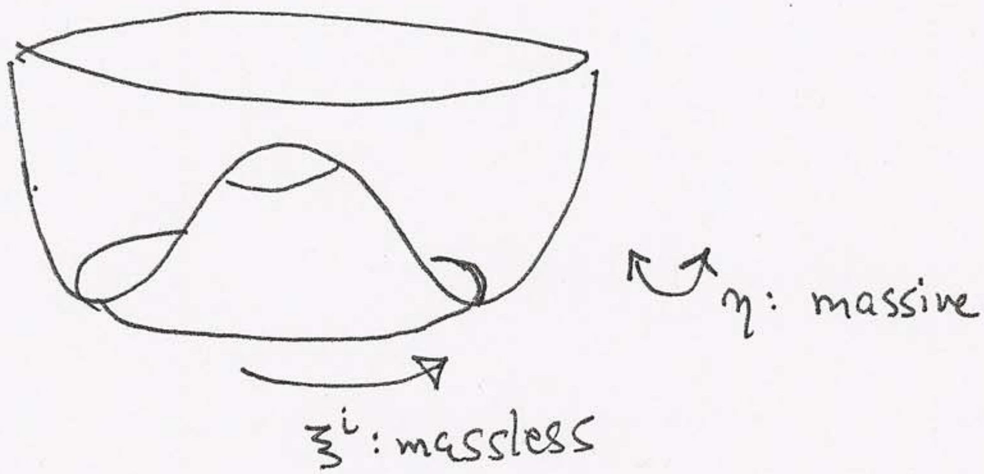
$\Rightarrow$ Minimum at $v = \sqrt{\frac{\mu^2}{\lambda}} \approx 246$ GeV $\Rightarrow \langle\phi\rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$

Parametrize Higgs in terms of deviations relative to new vacuum:

$$\langle\phi\rangle = \exp\left[-\frac{i\vec{\xi}(x)\cdot\vec{\tau}}{v}\right]\begin{pmatrix} 0 \\ \frac{v+\eta(x)}{\sqrt{2}} \end{pmatrix}$$

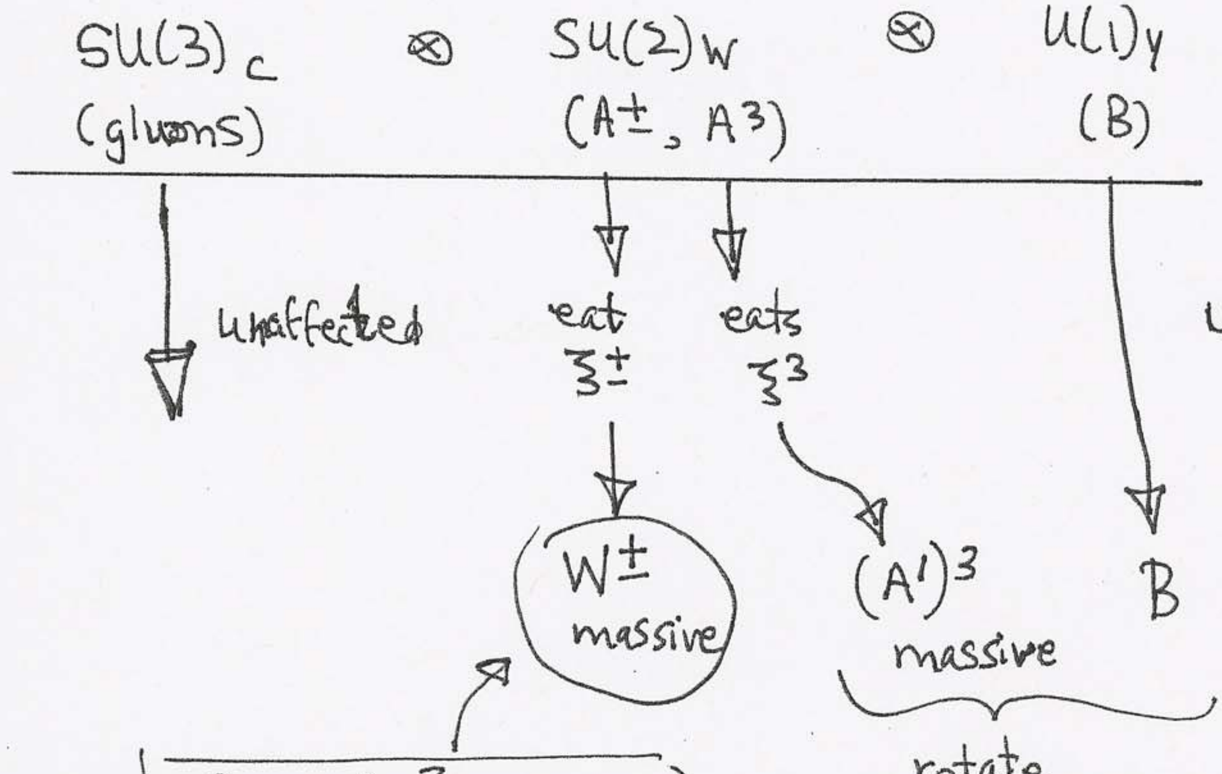
Thus, Higgs degrees of freedom are now

$$\begin{cases} \vec{\xi}(x): \ (\xi^{\pm}, \xi^{3}) & \text{would-be Goldstone bosons} \\ \eta(x): & \text{the physical Higgs} \end{cases}$$



So how does the Higgs mechanism actually work?

The Higgs Mechanism (schematically...)

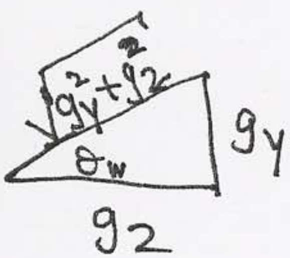


$$\boxed{\begin{aligned} M^2_{W^\pm} &= \frac{g_2^2 v^2}{4} \approx 83\,\text{GeV} \\ M^2_{Z^0} &= M_W^2/\cos^2\theta \approx 91\,\text{GeV} \end{aligned}}$$

$$\begin{pmatrix} Z^0 \\ \gamma \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} A'^3 \\ B \end{pmatrix}$$

where

$$\tan\theta_W = \frac{g_Y}{g_2} \Longrightarrow \sin^2\theta_W = \frac{g_Y^2}{g_Y^2 + g_2^2}$$



$$= \frac{\alpha_Y}{\alpha_Y + \alpha_2} \approx 0{,}23$$

Then

$$\left.\begin{aligned} \alpha_Y^{-1} &= \alpha_{EM}^{-1} \cos^2\theta_W \\ \alpha_2^{-1} &= \alpha_{EM}^{-1} \sin^2\theta_W \end{aligned}\right\} \Longrightarrow \alpha_{EM}^{-1} \approx 127$$

✓

and we have

$$\boxed{Q_{EM} = T_3 + \frac{Y}{2}}$$

$\boxed{\text{That's it!}}$

Well, not really...

1. Yukawa couplings!

$$\mathcal{L} \sim \boxed{y_d}\, \bar{Q}_L \phi\, d_R + \boxed{y_u}\, \bar{Q}_L (i\tau_2 \phi^*)\, u_R$$

$$+ \boxed{y_e}\, \bar{L}_L \phi\, e_R + \boxed{y_\nu}\, \bar{L}_L (i\tau_2 \phi^*)\, \nu_R + h.c.$$

only if neutrinos have Dirac masses

Then our fermions gain Dirac masses

$$m_i \doteq y_i \langle\phi\rangle$$

$$\Rightarrow \boxed{m_i = \frac{y_i v}{\sqrt{2}}}$$

(2) three generations!

essentially the fermion structure repeats,
but with one subtlety
— mixing between generations

QUARKS: $\begin{pmatrix} u \\ d' \end{pmatrix} \begin{pmatrix} c \\ s' \end{pmatrix} \begin{pmatrix} t \\ b' \end{pmatrix}$

where
$$\underset{\substack{SU(2) \\ \text{gauge} \\ \text{eigenstates}}}{\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}} = \underbrace{\left(\begin{array}{c|c} \text{cabibbo} & \\ \hline & \end{array}\right)}_{\text{CKM matrix}} \underset{\substack{\text{mass} \\ \text{eigenstates}}}{\begin{pmatrix} d \\ s \\ b \end{pmatrix}}$$

LEPTONS: $\begin{pmatrix} \nu_e' \\ e \end{pmatrix} \begin{pmatrix} \nu_\mu' \\ \mu \end{pmatrix} \begin{pmatrix} \nu_\tau' \\ \tau \end{pmatrix}$

where
$$\begin{pmatrix} \nu_e' \\ \nu_\mu' \\ \nu_\tau' \end{pmatrix} = \begin{pmatrix} 3\times 3 \\ \text{MNS} \\ \text{matrix} \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}$$

$\Rightarrow$ Yukawa couplings are matrices y_{AB} in flavor space

... flavor physics! CP violation, etc...
VERY COMPLICATED,
NO DEEP UNDERSTANDING!

STANDARD MODEL SUMMARY

$$SU(3)_c \otimes SU(2)_W \otimes U(1)_Y$$

gauge bosons spin-1	gluons :	$(8, \mathbb{1})_0$
	$A^{\pm}, A^3$:	$(\mathbb{1}, 3)_0$
	B :	$(\mathbb{1}, \mathbb{1})_0$
matter (all left-handed) spin-½	$Q = \begin{pmatrix} u \\ d \end{pmatrix}$:	$(3, 2)_{+1/3}$
	$L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}$:	$(\mathbb{1}, 2)_{-1}$
	u^c :	$(\bar{3}, \mathbb{1})_{-4/3}$
	d^c :	$(\bar{3}, \mathbb{1})_{+2/3}$
	e^c :	$(\mathbb{1}, \mathbb{1})_{+2}$
	[ν_e^c :	$(\mathbb{1}, \mathbb{1})_0$]
EW Higgs spin-0	ϕ :	$(\mathbb{1}, 2)_{+1}$

Question: $\boxed{\text{What sets values of } Y\text{?}}$

Note: $U(1)_Y$ is abelian group $\Rightarrow$ any normalizations are allowed!

For fermion (matter) content:

Since $Q_{EM} = T_3 + \frac{Y}{2}$

and since we have measured Q_{EM} experimentally, relative hypercharge assignments are fixed by experimental observations!

But is there a theoretical reason for these relative values of Y?

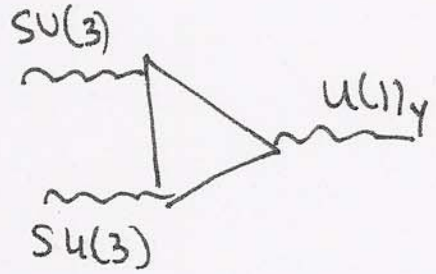
i.e., can we predict relations such as

$$|\text{proton charge}| = |\text{electron charge}|$$

on the basis of a physical principle?

$\boxed{\text{YES!}}$ Chiral (ABJ) triangle anomaly cancellation!
Anomalies spoil consistency of theory at the quantum level – come from diagrams

(1)



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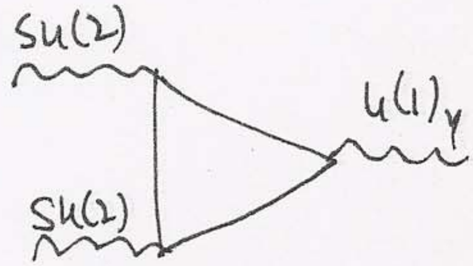

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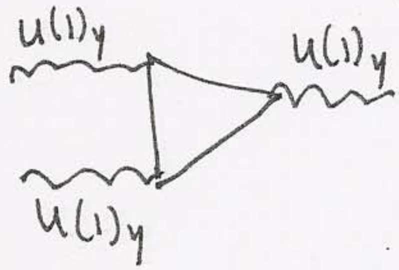

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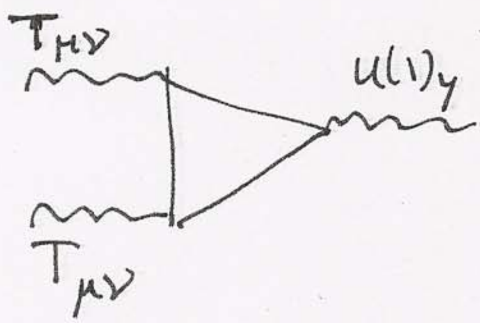


... where our chiral fermions run in the loops.

<u>Cancellation of each kind of anomaly diagram requires</u>

1. $\mathrm{Tr}_3 Y = 0$ summed over all <u>colored</u> fermions
2. $\mathrm{Tr}_2 Y = 0$ summed over all fermion <u>doublets</u>
3. $\mathrm{Tr}\, Y^3 = 0$ summed over all fermions with $Y \neq 0$
4. $\mathrm{Tr}\, Y = 0$ summed over <u>all</u> fermions

<u>UNIQUE SOLUTION IS THE SM Solution</u> (or its rescaling)

$\Rightarrow$ relative Y-values are fixed $\Rightarrow$ CHARGE QUANTIZATION!

but <u>overall</u> normalization still <u>unfixed</u>.

For Higgs field ϕ, situation is different.

Since ϕ not yet discovered,

we don't know its Q_{EM}.

However, assuming the same overall normalization as for the fermions,

we still have $\quad Q_{EM} = T_3 + Y/2$.

We then must choose

$$\phi : \quad (1, 2)_{+1} \quad \leftarrow \boxed{Y_\phi = +1}$$

So that

$$\phi = \begin{pmatrix} \phi^+ \\ \phi_0 \end{pmatrix} \begin{array}{l} \leftarrow Q_{EM} = +1 \\ \leftarrow Q_{EM} = 0 \end{array}$$

Why? Since bottom component ϕ_0 gets vev

i.e., $\langle \phi_0 \rangle = v/\sqrt{2}$,

it must be electrically neutral ($Q_{EM} = 0$)

so that EM is the remaining

unbroken symmetry !

STANDARD MODEL — Things to remember:

① lots of seemingly disconnected representations for gauge bosons & particle content

② three independent gauge couplings (g_3, g_2, g_Y)

— no predictions for g_i

— not even predictions for their ratios such as $\sin^2\theta_W = g_Y^2/(g_Y^2+g_2^2)$

③ particle representations are complex

e.g., $Q_L = (3,2)_{1/3}$ but no $(\bar{3},2)_{-1/3}$!

④ overall normalization for Y unfixed [since $U(1)_Y$ abelian]

even though relative Y-values are fixed

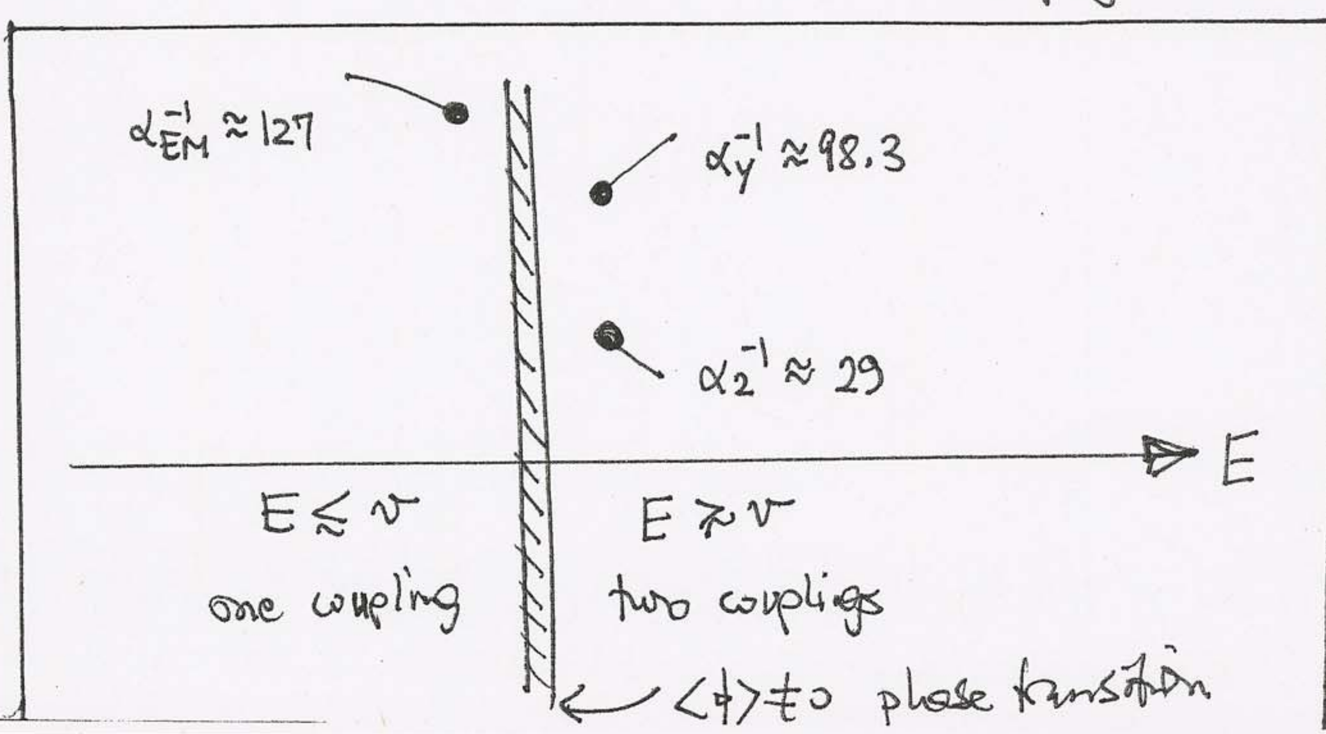
5. Higgs mechanism breaks

$$SU(2)_W \otimes U(1)_Y \xrightarrow{\langle\phi^0\rangle \neq 0} U(1)_{EM}$$

where ϕ^0 is EM-neutral.

In general, the subgroup which survives is the subgroup with respect to which the field getting the non-zero vev is neutral!

$$\sin^2\theta_W = \frac{g_Y^2}{g_Y^2 + g_2^2} \quad ; \quad M_{(W,Z)} \sim \underset{\text{coupling}}{g_2} \cdot \underset{\text{vev}}{v}$$



(6) <u>Particle representations</u> treat baryons & leptons <u>separately</u>

Also, they are not joined by <u>Yukawa couplings</u>

[Mixed Yukawa couplings would violate gauge invariance!]

$\Longrightarrow$ In SM,

- Baryon # (B) <u>conserved</u>
- Lepton # (L) <u>conserved</u>

Thus, e.g., the lightest baryon (= proton) is <u>STABLE</u>!

[<u>Note</u>: B is actually broken by instanton effects (<u>SMALL</u>)
L can be broken by ν Majorana mass terms (if they exist).]

② $\boxed{\text{Some group theory}}$

Let's think a bit more explicitly about the groups $\underline{SU(2)}$, $\underline{SU(3)}$ in the SM

$\Rightarrow$ What do they mean?

<u>Start with $SU(2) \approx SO(3)$ "rotation group"</u>

angular momentum!

Group has <u>3 generators</u>:

$$\begin{cases} \bullet \quad J_+ = J_1 + iJ_2 \\ \bullet \quad J_- = J_1 - iJ_2 \\ \bullet \quad J_z = J_3 \end{cases}$$

with commutation relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k \iff \begin{cases} [J_3, J_\pm] = \pm J_\pm \\ [J_+, J_-] = 2J_z \end{cases}$$

$\rightarrow$ Generators don't commute $\Rightarrow$ group is "non-abelian"

$\Rightarrow$ only one generator can be diagonalized

$\Rightarrow$ choose it to be J_z

Then states can be chosen as eigenstates of J_z

$$J_z|m\rangle = m|m\rangle$$

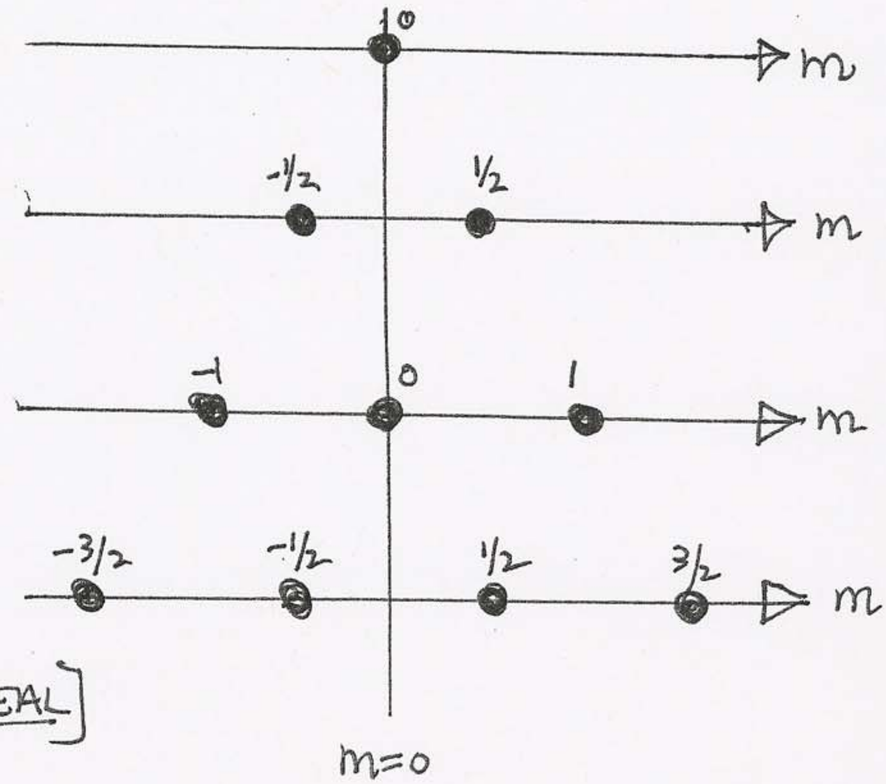
$\Rightarrow$ Can indicate states graphically in terms of m:

SINGLET REP. $\rightarrow$ "$j=0$"

SPINOR REP. $\rightarrow$ "$j=1/2$"
"fundamental"

VECTOR/ADJOINT $\rightarrow$ "$j=1$"

$\vdots$ "$j=3/2$"

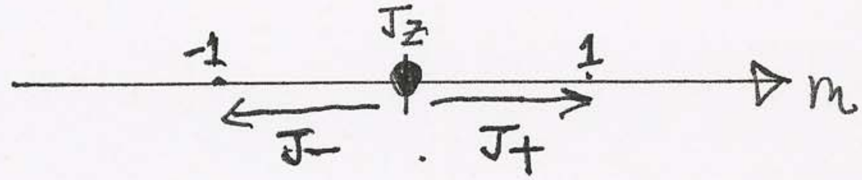


[ALL REPRESENTATIONS ARE REAL]

For each representation (j),
dimension of rep $= 2j+1$

$J_+|m\rangle \sim |m+1\rangle \Rightarrow J_+$ raises m by 1
$J_-|m\rangle \sim |m-1\rangle \Rightarrow J_-$ lowers m by 1
$J_z|m\rangle = m|m\rangle \Rightarrow J_z$ preserves m, gives location

Actions are graphically represented as:

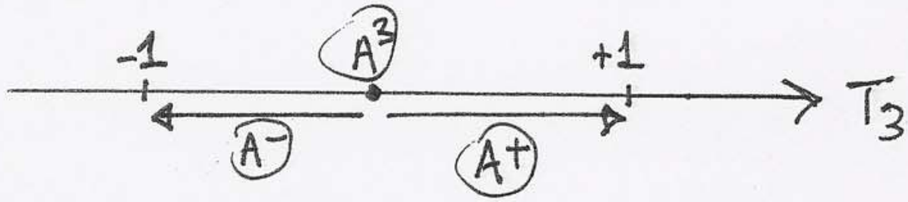


nothing
here -
don't copy

In the actual SM case of $SU(2)_W$,

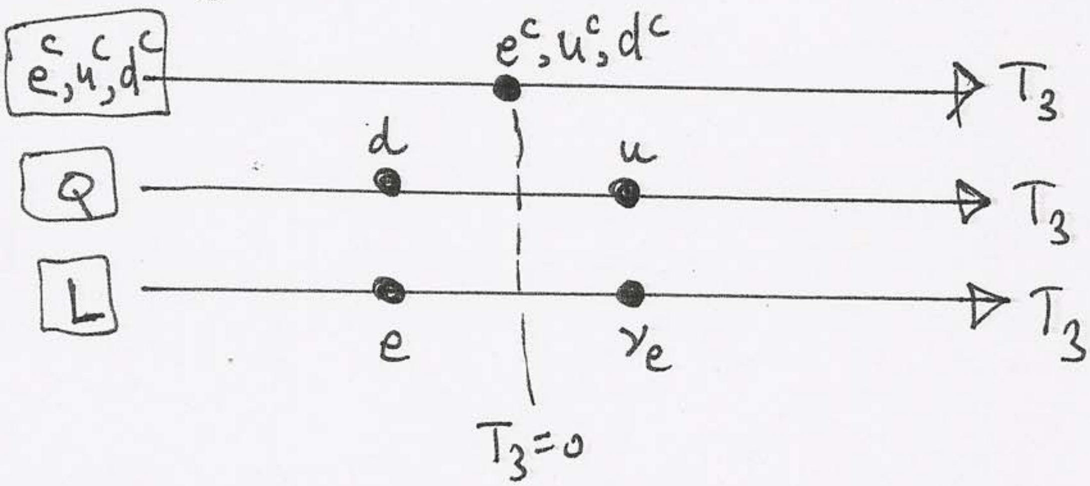
$$\begin{cases} m \to T_3 \\ J^{\pm} \to A^{\pm} \\ J^{z} \to A^{3} \end{cases}$$

Thus, the gauge bosons of $SU(2)_W$ have the actions

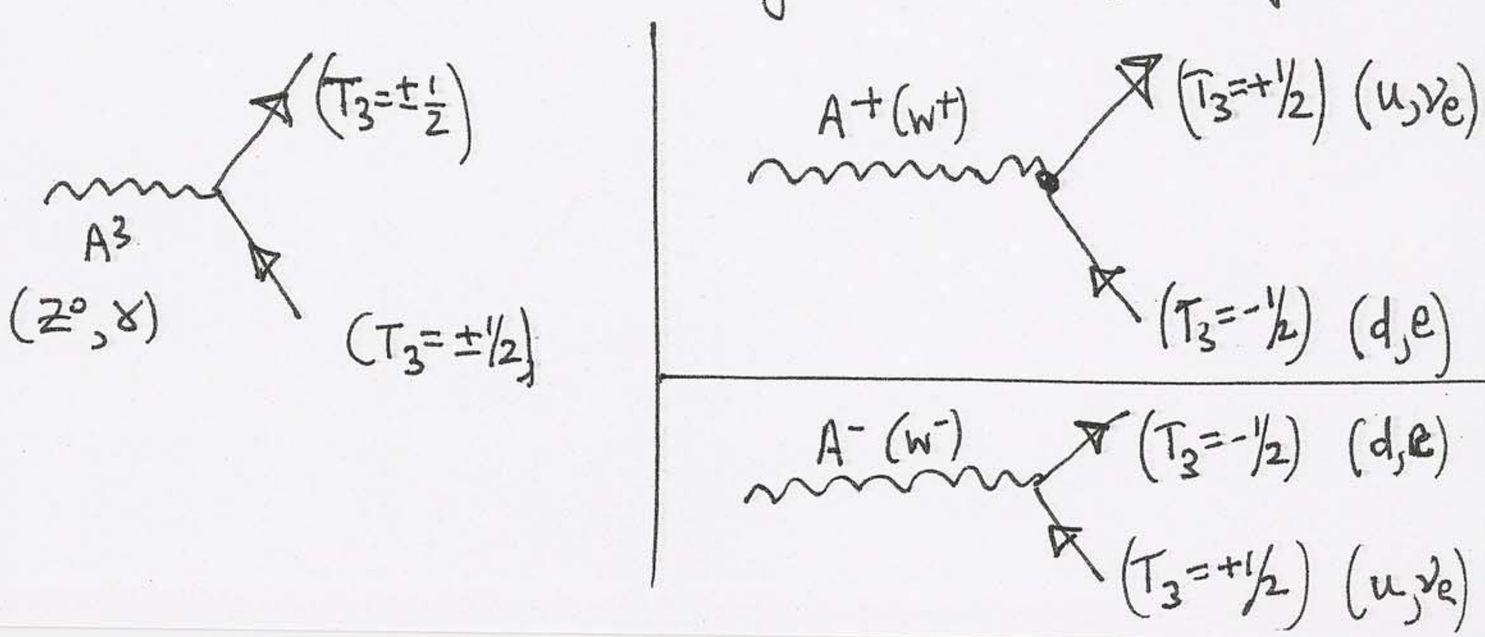


Moreover, as we saw, all fermion representations in the SM are either singlets or doublets:

e.g.,

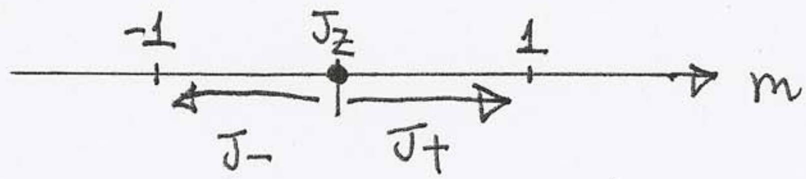


So gauge bosons act according to these T_3 "charges":



SU(2) SUMMARY

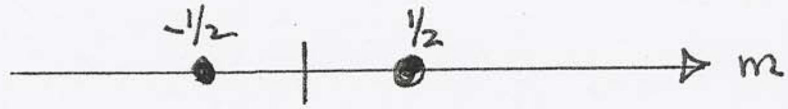
- <u>one</u> diagonal generator: $J_z|m\rangle = m|m\rangle$
- <u>three</u> generators total $(J_\pm, J_z)$:



- <u>representations</u>:

singlet

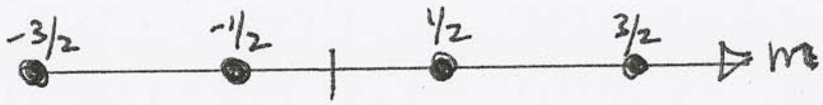

1

doublet


2

triplet


3

quadruplet


4

⋮

etc.

- all reps are <u>real</u> (i.e., same if inverted through origin, $m \to -m$)
- generator diagram is <u>same</u> points as <u>triplet</u> representation
  - $\Rightarrow$ triplet is "<u>adjoint</u>" representation.

Now consider $\boxed{SU(3)}$

↳ examples: $SU(3)_{color}$ (r, g, b) (LOCAL)

$SU(3)_{flavor}$ (u, d, s) (GLOBAL)

very different physically, but exactly the same algebraically ⟹ We shall use both examples!

SU(3): has eight generators $T^1, T^2, \ldots, T^8$

e.g., in color case ⟹ 8 gluons

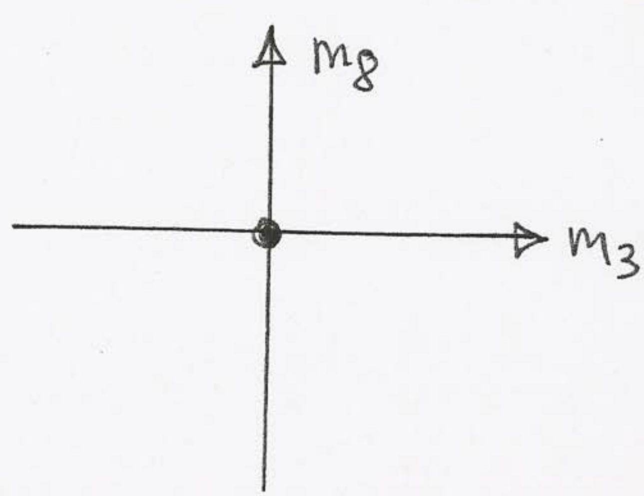
⟹ 8 Gell-Mann matrices $\lambda^{i=1,\ldots,8}$

ONLY TWO CAN BE SIMULTANEOUSLY DIAGONALIZED ⟹ usually called T^3, T^8

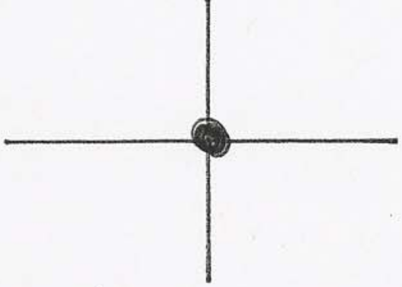
⟹ all states can be chosen as eigenstates of T^3, T^8:

$$T^3 |m_3, m_8\rangle = m_3 |m_3, m_8\rangle$$
$$T^8 |m_3, m_8\rangle = m_8 |m_3, m_8\rangle$$

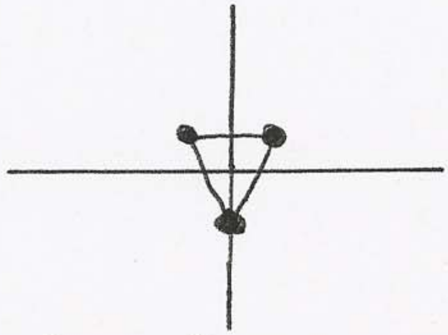
States can now be represented graphically in an (m_3, m_8) PLANE:



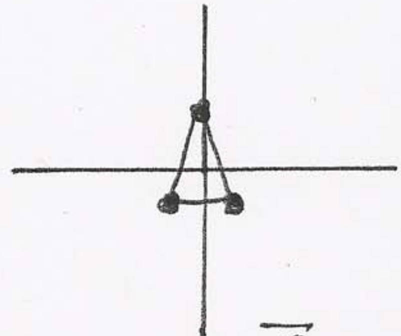
Just as for SU(2), only certain representations are allowed for self-consistency:



"singlet" $\mathbb{1}$

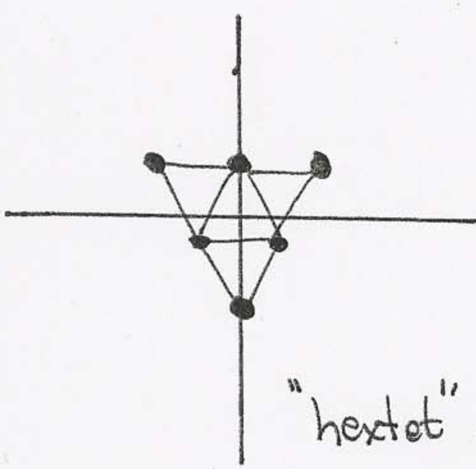


"triplet" $\mathbb{3}$
(fundamental)

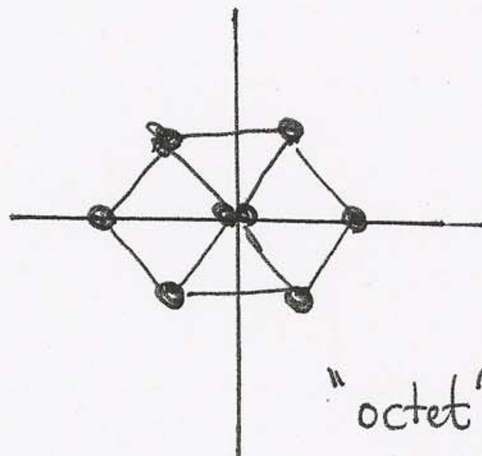


"antitriplet" $\overline{\mathbb{3}}$

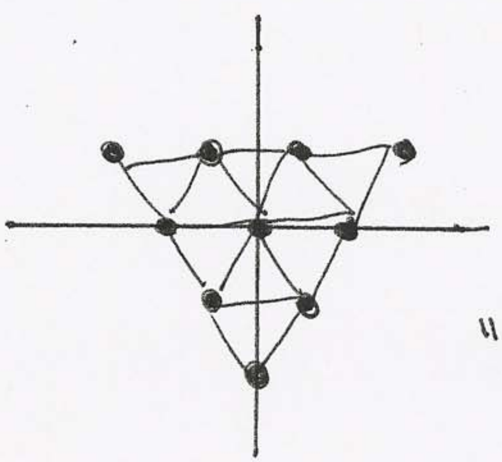
(<u>conjugate</u>: flip each point through origin!)



"hextet" $\mathbb{6}$



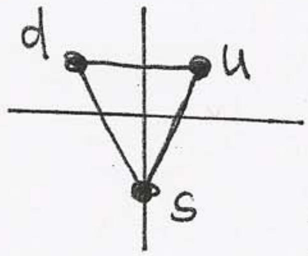
"octet" $\mathbb{8}$
(<u>real</u>!)
$8 = \overline{8}$



"decuplet" $\mathbb{10}$

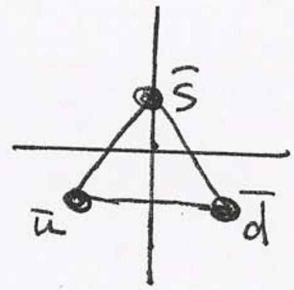
..., etc.

e.g., in flavor case, we have



$\mathbf{3}$

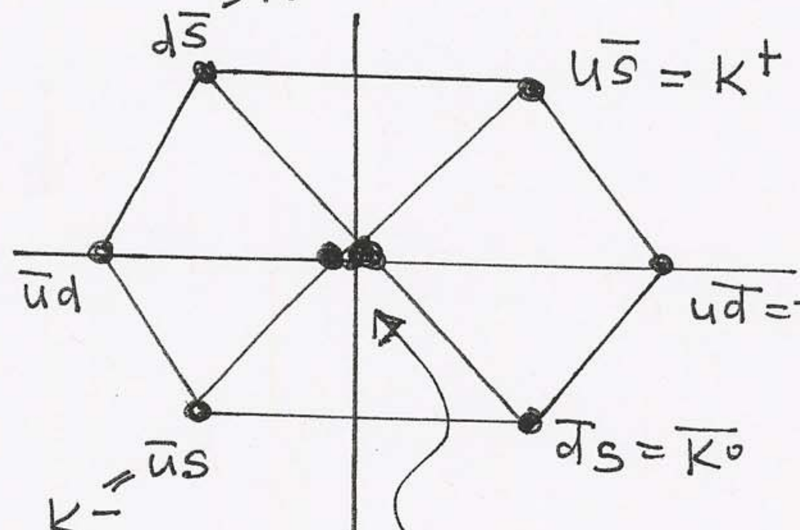
quarks



$\bar{\mathbf{3}}$

anti-quarks

Thus, "adding" these together in various combinations (i.e., taking the tensor product $\mathbf{3} \times \bar{\mathbf{3}}$) we obtain



"meson octet"

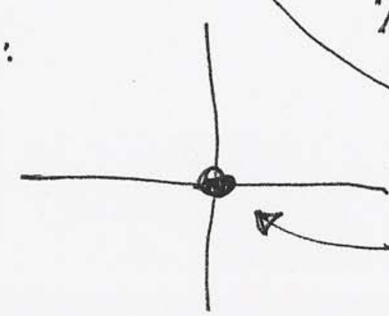
center locations are

$\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$

$\eta^0 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$

["SU(3) Clebsch-Gordon coefficients"]

plus a singlet:



$u\bar{u} + d\bar{d} + s\bar{s}$

It turns out that <u>all</u> larger reps can be constructed in this way by tensoring together combinations of the fundamental representations:

- e.g. $\quad 3 \otimes \bar{3} = 8 \oplus \mathbb{1}$ MESONS (as we just saw)

$$3 \otimes 3 = 6 \oplus 3$$

$$6 \otimes 3 = 10 \oplus 8$$

$$3 \otimes 3 \otimes 3 = 10 + 8 + 8 \oplus \mathbb{1} \quad \text{BARYONS}$$

Big representations are just products of small representations!

These relations can also be represented by <u>matrices</u>.

Imagine the fundamental rep as a vector

$$3_i = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \; ; \quad \bar{3}_{\bar{i}} = (\bar{u}\,\bar{d}\,\bar{s})$$

Then, e.g.,

- $$3_j \otimes 3_i = \begin{array}{c} \xrightarrow{i=(uds)} \\ (M_{ji}) \\ \big\downarrow\, j=(uds) \end{array} = \underset{\text{symmetric}}{S_{ji}} \oplus \underset{\text{anti-symmetric}}{A_{ji}}$$
$$= 6_{ji} \oplus \bar{3}_{ji}$$

- $$\bar{3}_{\bar{j}} \otimes 3_i = \begin{array}{c} \xrightarrow{i=(uds)} \\ M_{\bar{j}i} \\ \big\downarrow\, \bar{j}=(\bar{u}\bar{d}\bar{s}) \end{array} = \underset{\text{traceless}}{\tilde{M}_{\bar{j}i}} \oplus \underset{\text{trace}}{T}$$
$$= 8 \oplus 1$$

where

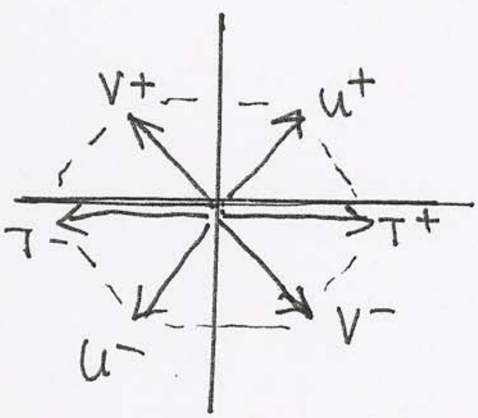
$$8 = \begin{bmatrix} \bar{u}u & \bar{u}d & \bar{u}s \\ \bar{d}u & \bar{d}d & \bar{d}s \\ \bar{s}u & \bar{s}d & \bar{s}s \end{bmatrix} - \frac{1}{3}\left(u\bar{u} + d\bar{d} + \bar{s}s\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

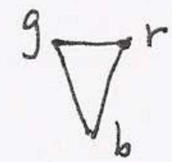
$$\rightleftharpoons \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta^0 & \pi^+ & K^+ \\ \pi^- & \frac{-\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -2\eta^0/\sqrt{6} \end{pmatrix}$$

For $SU(3)$, there are 8 generators:

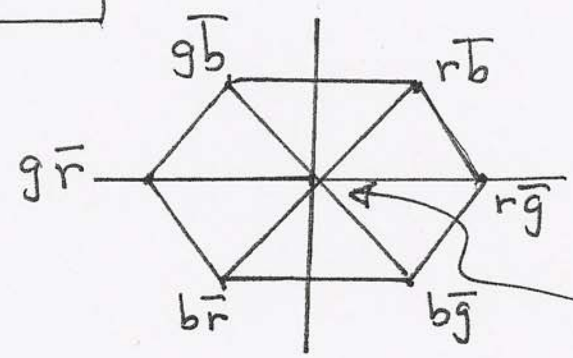
- T^3, T^8 $\leftarrow$ analogues of J_z for $SU(2)$
- $\underbrace{\text{six others}}$ $\leftarrow$ analogues of $J_\pm$

these do "raising" and "lowering" in the two-dimensional (m_3, m_8) plane:



For color 

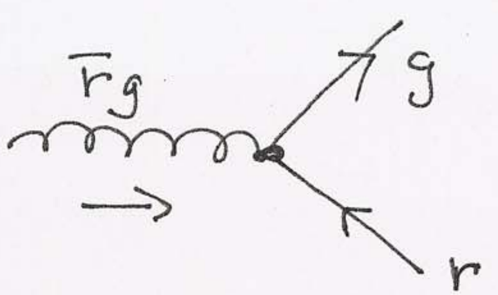
these generators are all gluons

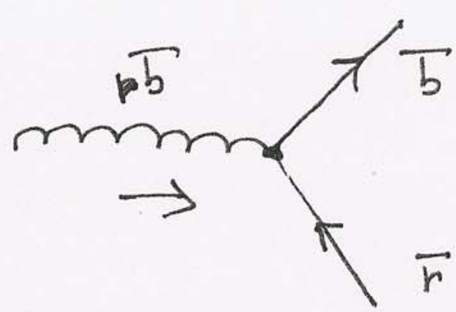


$$T^3 = \frac{r\bar{g} - g\bar{r}}{\sqrt{2}}$$

$$T^8 = \frac{r\bar{r} + g\bar{g} - 2b\bar{b}}{\sqrt{6}}$$

gluons act on quark lines as we expect, e.g.,





$\boxed{\text{Aside}}$: Why is it called SU(3)?

We have already seen that the generators are all

(3×3)-dimensional traceless Hermitian matrices

$$
\begin{array}{c}
\xrightarrow{\qquad\qquad\qquad} \; b=(rgb) \\
(T^i)_{\bar{a}b} = \begin{pmatrix} r\bar{r}-T/3 & \bar{r}g & \bar{r}b \\ \bar{g}r & g\bar{g}-T/3 & \bar{g}b \\ \bar{b}r & \bar{b}g & \bar{b}b-T/3 \end{pmatrix} \\
\downarrow \; \bar{a}=(\bar{r}\bar{g}\bar{b})
\end{array}
$$

where: $T = \bar{r}r + \bar{g}g + \bar{b}b$

These Hermitian generators are like operators $\vec{J}$

Recall that such operators can be exponentiated to form group elements

for SU(2): e.g., $D(\phi, \hat{n}) = \exp\left[i\vec{J}\cdot\hat{n}\,\delta\phi\right]$

unitary rotation operator!

In general, group elements are

$$U(\epsilon_a) = \exp\left[i\sum_a \epsilon_a T_a\right]$$

ϵ_a: parameters
T_a: generators

Since T^a are Hermitian & traceless (3×3)

$\Rightarrow$ U are unitary with det = 1 (3×3)

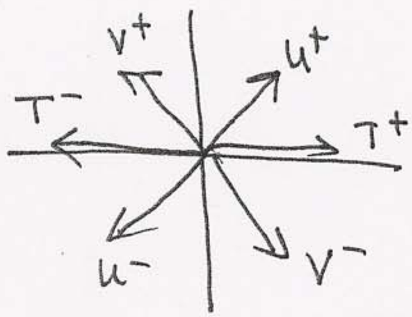
$\Rightarrow$ SU(3)!

SU(3) summary :

- two diagonal generators

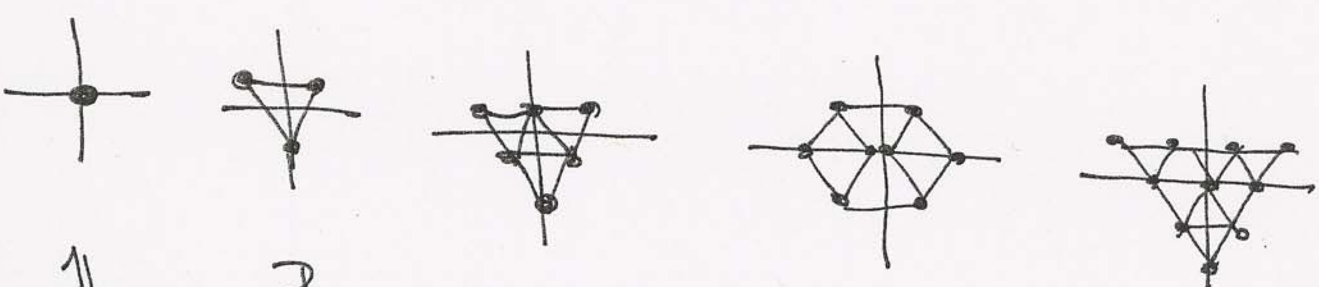
$$T_3 |m_3, m_8\rangle = m_3 |m_3, m_8\rangle$$
$$T_8 |m_3, m_8\rangle = m_8 |m_3, m_8\rangle$$

- together, all generators fill out the pattern



with T^3, T^8 in the center (do not raise or lower)

- SU(3) representations are



$\mathbb{1}$ 3   6   8   10

etc.

Note : $\mathbb{1}$, 8 are real

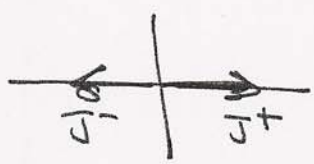
3, 6, 10 are complex ...

(8 is adjoint)

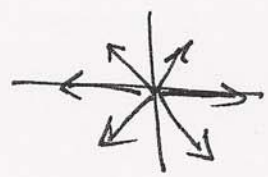
In general, these types of patterns continue to LARGER GROUPS as well.

In each case,

- \# of commuting generators (like T^3, T^8)
  $\equiv$ dimensionality of our plots
  $\equiv$ RANK of group
- total \# of generators $\equiv$ ORDER of group
- generators fill out pictures



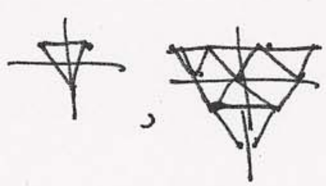
SU(2)



SU(3)

} these vectors are called "roots".

- representations are states which fill out dot patterns



, etc.

} These dots are called "weights".

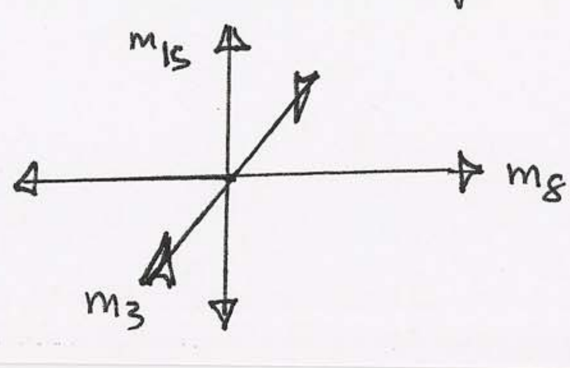
We can then classify all possible groups!

COMPLETE CLASSIFICATION OF LIE GROUPS:

name	rank	serial number (order)
$SU(n)$	$n-1$	n^2-1
$SO(2n+1)$	n	$n(2n+1)$
$Sp(2n)$	n	$n(2n+1)$
$SO(2n)$	n	$n(2n-1)$
E_6	6	78
E_7	7	133
E_8	8	248
F_4	4	52
G_2	2	14

e.g., $SU(4)$: rank 3; 15 generators $(\underbrace{T^3, T^8, T^{15}}_{\text{diagonal}})$

plot representations
in a 3-dimensional space:



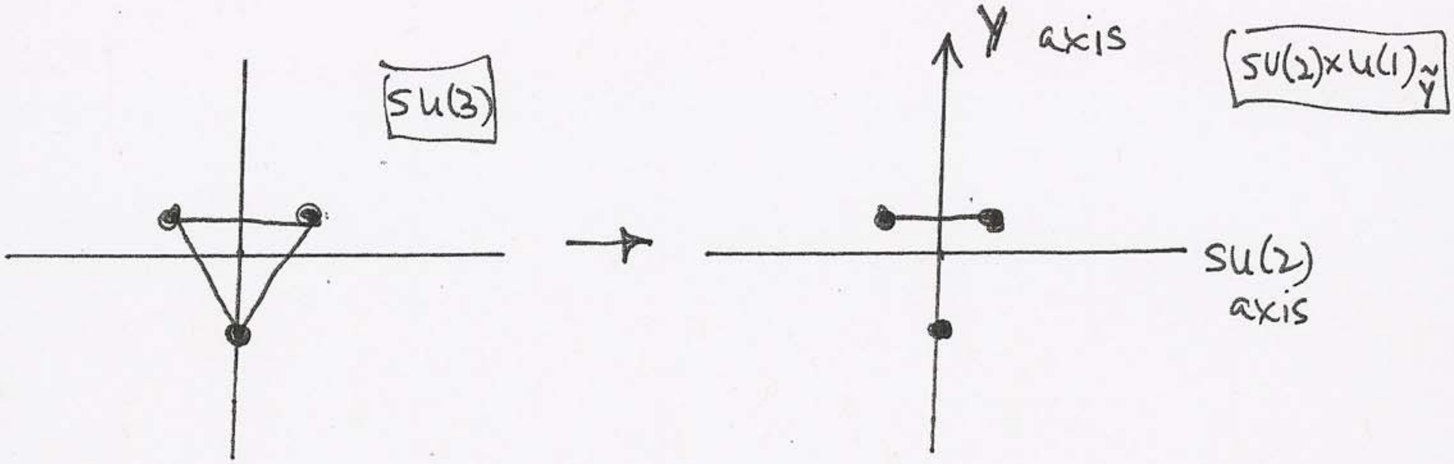
$\boxed{\text{SUBGROUPS}}$

e.g., $SU(3) \to SU(2) \otimes U(1)$

"flavor"   "isospin" $\tilde{Y}$

[$\tilde{Y}$ = weak isospin
= strangeness + const].

Just "decompose" representations!



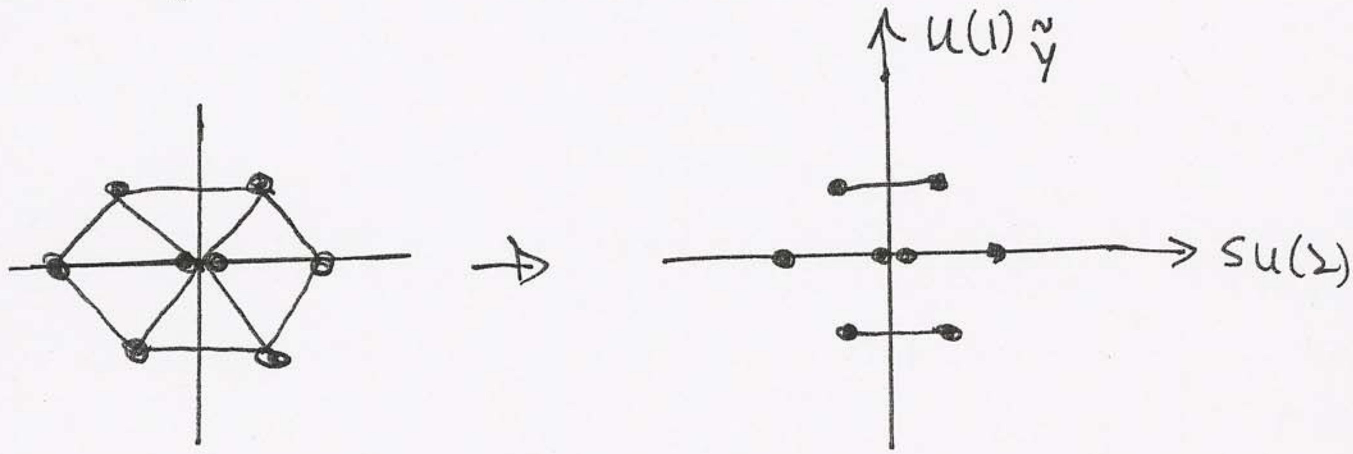
$$3 \longrightarrow (2)_{\frac{1}{2\sqrt{3}}} \oplus (1)_{-1/\sqrt{3}}$$

We can rescale $\tilde{y}$ to eliminate fractions if we wish:

$$3 \longrightarrow (2)_1 \oplus (1)_{-2}$$

$\boxed{\text{TWO REPRESENTATIONS}}$ OF SUBGROUP, $SU(2)\otimes U(1)$,
WHEN PROPERLY CHOSEN,
COMBINE INTO A $\boxed{\text{SINGLE REPRESENTATION}}$ OF
A BIGGER GROUP $SU(3)$!

eg, adjoint rep of $SU(3)$:



$$8 \longrightarrow (2)_{+1} \oplus (2)_{-1} \oplus (3)_0 \oplus (1)_0$$

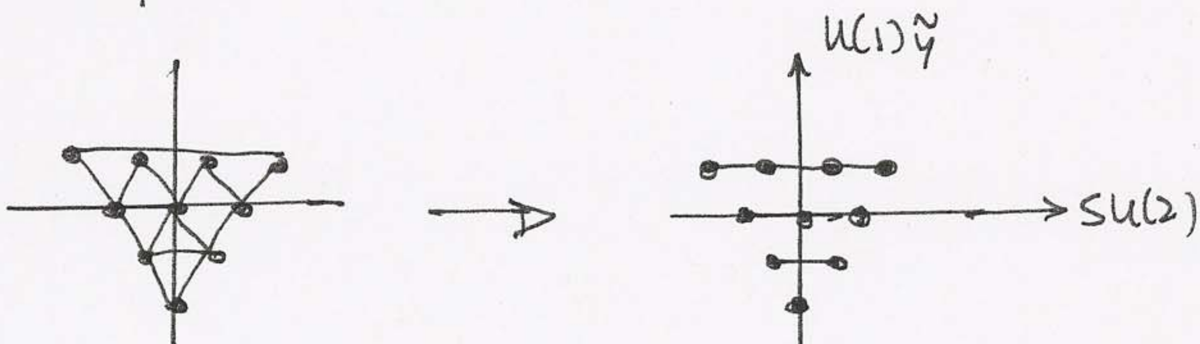
real

conjugates

real real

✓

eg, decuplet



$$10 \longrightarrow (4)_{+1} \oplus (3)_0 \oplus (2)_{-1} \oplus (1)_{-2}$$

If H is a subgroup of G,
then <u>every</u> rep of G can
decompose into sums of reps of H!

Or, going backwards, SOMETIMES

a set of reps of one group,
when properly chosen with all quantum numbers properly balanced, can combine to fill out a

SINGLE REP of a LARGER GROUP !

Hmmm...

That was our original goal!
– to "unify" all of the forces and particles!

We now see how to do this:

$\Rightarrow$ We need a bigger group!

(and hope for a few miracles along the way...)

What groups G can we choose?

Requirements:

① SM has rank = 4 (four commuting generators) $\underbrace{T^3, T^8}_{\text{gluons}}$ $\underbrace{A^3, B}_{Z^0, \gamma}$ $\Rightarrow$ group G must be rank ≥ 4 and contain SM as subgroup

② SM has complex representations [eg $(3,2)_{1/3}$, no $(\bar{3},2)_{-1/3}$] $\Rightarrow$ group G must also have complex reps (eg SU(2) doesn't, SU(3) does)

③ SM is free of chiral anomalies $\Rightarrow$ group G must have reps for which anomalies can cancel

④ If we want to relate the couplings (g_Y, g_2, g_3) to each other $\Rightarrow$ G should be a simple group (not a product of different, unrelated factors)

Look back at previous list of groups.
What are our options?

rank 1	U(1) SU(2)				
rank 2	SU(3)	SO(5)			G_2
rank 3	SU(4)	SO(7)	Sp(6)		
rank 4	SU(5)	SO(9)	Sp(8)	SO(8)	F_4
rank 5	SU(6)	SO(11)	Sp(10)	SO(10)	
rank 6	SU(7)	SO(13)	Sp(12)	SO(12)	E_6
rank 7	SU(8)	SO(15)	Sp(14)	SO(14)	E_7
rank 8	SU(9)	SO(17)	Sp(16)	SO(16)	E_8
	⋮	⋮	⋮	⋮	

CIRCLE INDICATES GROUP HAS COMPLEX REPS.

Therefore, our options are:

- if G = simple, then $G = SU(5)$ rank 4
  $SU(6)$, $SO(10)$ rank 5
  $SU(7)$, E_6 rank 6

- if G = product group, then
  - require complex factor to contain $SU(3)$ as subgroup
  - remaining factors to contain $SU(2) \otimes U(1)$

$\Rightarrow$ $G =$

$$\left.\begin{array}{l} SU(3) \times SU(2) \times SU(2) \\ SU(3) \times SU(3) \\ SU(3) \times SO(5) \end{array}\right] \text{rank } 4$$

$$\left.\begin{array}{l} SU(4) \times SU(2) \times U(1) \\ SU(4) \times SU(2) \times SU(2) \\ SU(4) \times SU(3) \\ SU(4) \times SO(5) \\ SU(3) \times SO(7) \\ SU(3) \times Sp(6) \end{array}\right] \text{rank } 5$$

$\vdots$ etc.

Most of these choices do not succeed in producing interesting unifications.

<u>However, interesting cases are:</u>

rank - 4 : $SU(5)$ Georgi-Glashow

rank - 5 : $\begin{cases} SO(10) \\ SU(4) \times SU(2) \times SU(2) \end{cases}$ Pati-Salam

rank - 6 : E_6

<u>These groups have the relative subgroup structure:</u>

$$\begin{tikzcd}[column sep=small]
\boxed{E_6} \arrow[r] & \boxed{SO(10)} \otimes U(1) \arrow[dl] \arrow[dr] & & \text{rank 6} \\
\boxed{SU(5)} \otimes U(1) \arrow[dr] & & \boxed{SU(4) \otimes SU(2) \otimes SU(2)} \arrow[dl, "\otimes U(1)"] & \text{rank 5} \\
& \boxed{SU(3) \otimes SU(2) \otimes U(1)} & & \text{rank 4}
\end{tikzcd}$$

Let's begin by looking at $SU(5)$.

$\boxed{SU(5)} \to SU(3) \otimes SU(2) \otimes U(1)$

Representations decompose as

$$\mathbb{1} \to (\mathbb{1},\mathbb{1})_0$$
$$5 \to (3,\mathbb{1})_{-2} \oplus (\mathbb{1},2)_3$$
$$10 \to (3,2)_1 \oplus (\bar{3},\mathbb{1})_{-4} \oplus (\mathbb{1},\mathbb{1})_6$$
$$15 \to (6,\mathbb{1})_4 \oplus (3,2)_1 \oplus (\mathbb{1},3)_6$$
$$24 \to (8,\mathbb{1})_0 \oplus (3,2)_{-5} \oplus (\bar{3},2)_5 \oplus (\mathbb{1},3)_0 \oplus (\mathbb{1},\mathbb{1})_0$$

30, 40,
45, 50, 70 etc...

where all $U(1)$ charges are normalized to avoid fractions
(as is conventionally done in standard references...)

<u>Does this have the potential for a successful unification?</u>

① <u>Fermions of SM:</u>

Recall each SM generation contains 15 states

$$\underbrace{\begin{pmatrix} u \\ d \end{pmatrix} \; \begin{matrix} (u^c) \\ (d^c) \end{matrix}}_{\times 3 \text{ colors}} \qquad \begin{pmatrix} \nu_e \\ e \end{pmatrix} \; (e^c) \quad = 15 \text{ states}$$

However, the $\mathbb{15}$ representation of $SU(5)$ does <u>not</u> accommodate them

(e.g., $\mathbb{15}$ gives a color <u>sextet</u>!)

[Nor does the $\mathbb{45}$ representation accommodate the <u>three</u> generations.]

<u>But look at $\mathbb{10}$</u> – if we rescale above $U(1)$ quantum #'s by $1/3$, it becomes

$$\mathbb{10} \longrightarrow \underbrace{(\mathbb{3}, \mathbb{2})_{+1/3}}_{Q} \oplus \underbrace{(\bar{\mathbb{3}}, \mathbb{1})_{-\frac{4}{3}}}_{u^c} + \underbrace{(\mathbb{1}, \mathbb{1})_{2}}_{e^c}.$$

All that's left is
$$\left.\begin{aligned} d^c &= (\bar{\mathbb{3}}, 1)_{2/3} \\ L &= (1, 2)_{-1} \end{aligned}\right\} \text{ 5 states}$$

$\Rightarrow$ These don't fit into a $\mathbb{5}$

but into a $\bar{\mathbb{5}}$!

Thus, an entire SM generation fits into

$$\boxed{\quad \bar{\mathbf{5}} \oplus \mathbf{10} \quad \text{of } SU(5) \quad}$$

with nothing left over (no exotics)!

In matrix notation, this is

$$\bar{\mathbf{5}} = \left(\underbrace{d_r^c \ d_g^c \ d_b^c}_{\text{color part}} \;\vdots\; \underbrace{\nu_e \ e}_{\text{weak part}} \right) \qquad \text{row vector}$$

$\mathbf{10}$ = antisymmetric component of 5×5 matrix

$$= \left(\begin{array}{c|c} u^c & Q \\ \hline & e^c \end{array} \right) = \left(\begin{array}{ccc|cc} 0 & u_r^c & u_g^c & u_r & d_r \\ & 0 & u_b^c & u_g & d_g \\ & & 0 & u_b & d_b \\ \hline & & & 0 & e^c \\ & & & & 0 \end{array} \right)$$

where (recall) $\begin{cases} u^c : (\bar{3}, 1) & : SU(3) \text{ triplet}, \ SU(2) \text{ singlet} \\ Q : (3, 2) : & \text{triplet}, \quad \text{doublet} \\ e^c : (1, 1) & \text{singlet}, \quad \text{singlet} \end{cases}$

② Gauge bosons must come from adjoint rep:

$$24 \rightarrow \underbrace{\underbrace{(8,1)_0}_{\text{gluons}} \oplus \underbrace{(1,3)_0}_{A^\pm,\, A^3} \oplus \underbrace{(1,1)_0}_{B}}_{} \oplus \underbrace{(3,2)_{-\frac{5}{3}} \oplus (\bar{3},2)_{\frac{5}{3}}}_{}$$

SM gauge bosons
all successfully embedded !

$\left[\text{Proof that } SU(3) \otimes SU(2) \otimes U(1) \subset SU(5)\ \checkmark\right]$

BUT WHAT ARE THESE ??!

Appear to be gauge bosons carrying

- color
- Weak charge
- hypercharge ... simultaneously !

Also,
$$Q_{EM} = T_3 + \frac{Y}{2} = \left\{ \pm\frac{1}{3}, \pm\frac{4}{3} \right\}$$

They are also electromagnetically fractionally charged as well !

In matrix language:

since $\mathbf{5} \otimes \bar{\mathbf{5}} = \underbrace{\mathbf{24}}_{} \oplus \mathbf{1}$

This is a traceless matrix:

$$
\mathbf{24} = \left(\begin{array}{ccc|cc}
 & & & X_r & Y_r \\
 & \text{(gluons)} & & X_g & Y_g \\
 & & & X_b & Y_b \\
\hline
X_{\bar{r}} & X_{\bar{g}} & X_{\bar{b}} & & \\
Y_{\bar{r}} & Y_{\bar{g}} & Y_{\bar{b}} & \multicolumn{2}{c}{(A^{\pm}, A^3, B)}
\end{array}\right)
$$

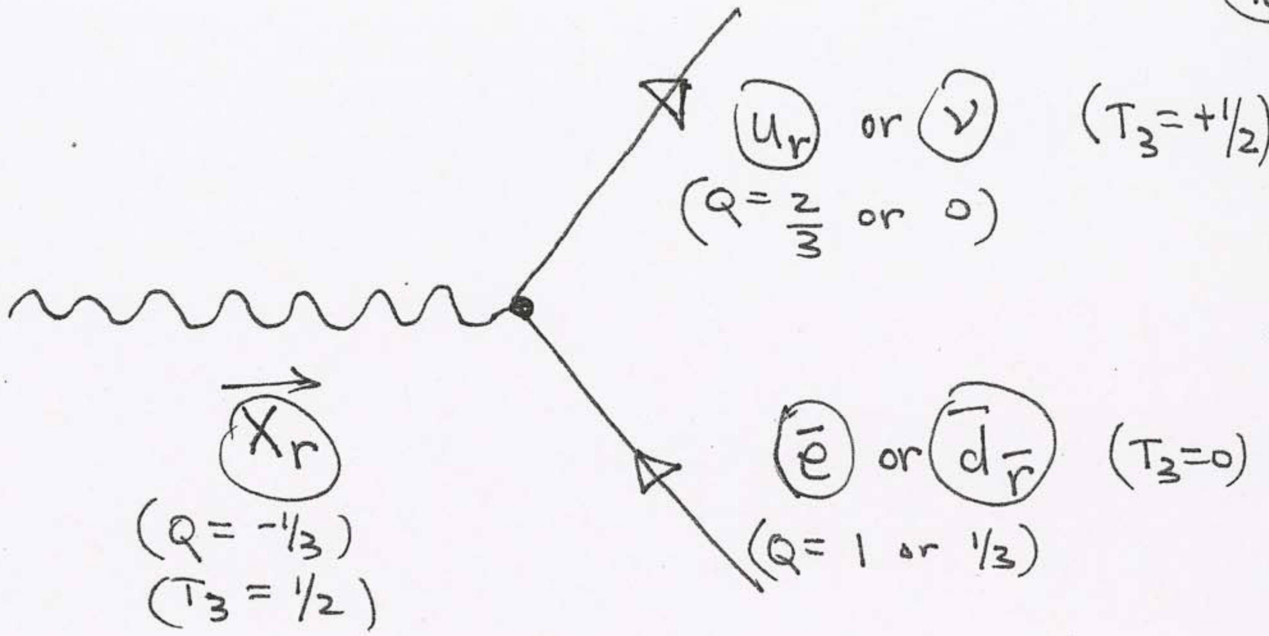
$Q_{EM}^{(X)} = -1/3$
$Q_{EM}^{(Y)} = -4/3$

$Q_{EM}^{(X)} = +1/3$
$Q_{EM}^{(Y)} = +4/3$

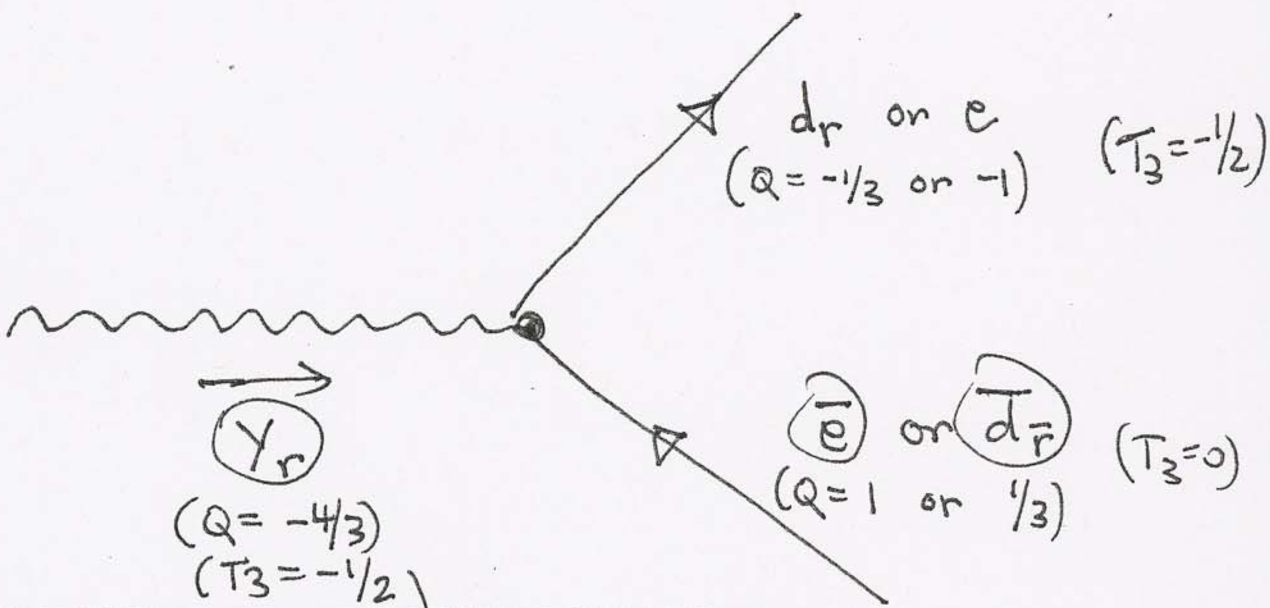
$\Rightarrow$ "X, Y" gauge bosons are "off-diagonal" in color/electroweak space, carry both types of charges simultaneously!

$\Rightarrow$ They can connect quarks to leptons!

e.g.,



or



- In each case $\left\{ \begin{array}{l} Q_{EM} \text{ is conserved} \\ T_3 \text{ is conserved} \\ \text{color lines flow correctly} \end{array} \right.$

X, Y appear as "leptoquarks" – injecting color and $T_3 = \pm 1/2$!

$\Rightarrow$ Baryon # and lepton # no longer conserved!

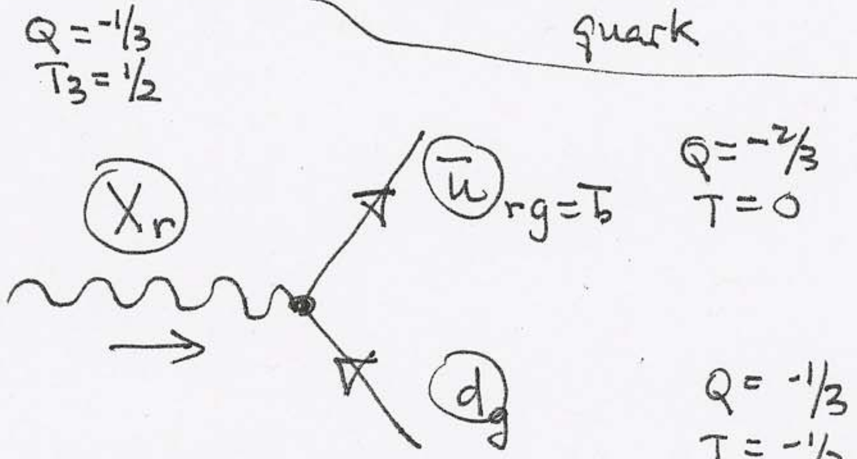
Each of these diagrams is $\boxed{\Delta B = 1/3}$ process!
("LEPTOQUARK CHANNEL")

There is also another channel in which (X, Y) bosons can act:

Since (X,Y) carry $\underline{3}$ of color, they can also turn quarks directly into antiquarks because

$$\underset{\text{from}(X,Y)}{3} \otimes \underset{\text{from quark}}{3} = 6 \oplus \underbrace{\bar{3}}_{\text{becomes antiquark}}$$

eg



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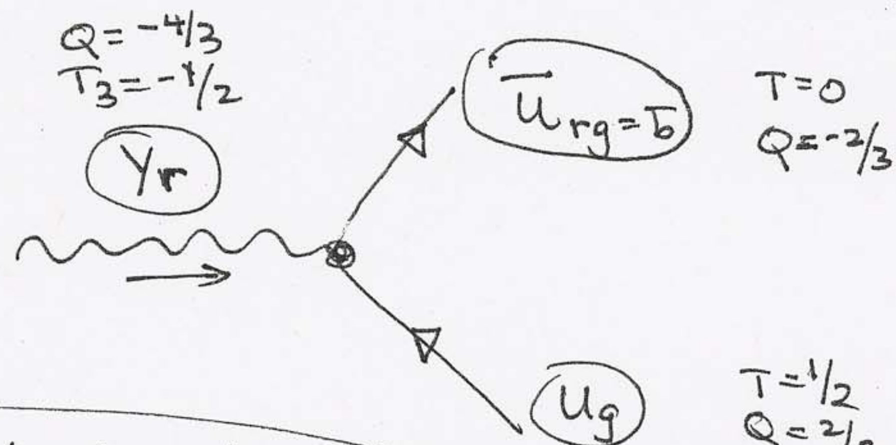


In such channels, (X,Y) act as "di-quarks", not leptoquarks!

Here $\boxed{\Delta B = -2/3}$ since $q \to \bar{q}$.

Note that in each channel,

(X, Y) gauge bosons change

<u>incoming particles</u>

into

<u>outgoing antiparticles</u> !

Very strange...

<u>fermion # is not conserved</u>!

3 Finally, the Higgs representation $\phi : (\mathbb{1}, 2)_{+1}$ is embedded easily into the $\underline{5}$ rep of SU(5):

$$\underline{5} \to \underbrace{(3, \mathbb{1})_{-2/3}} \oplus (\mathbb{1}, 2)_1$$

↳ usual Higgs doublet ϕ ✓

↓

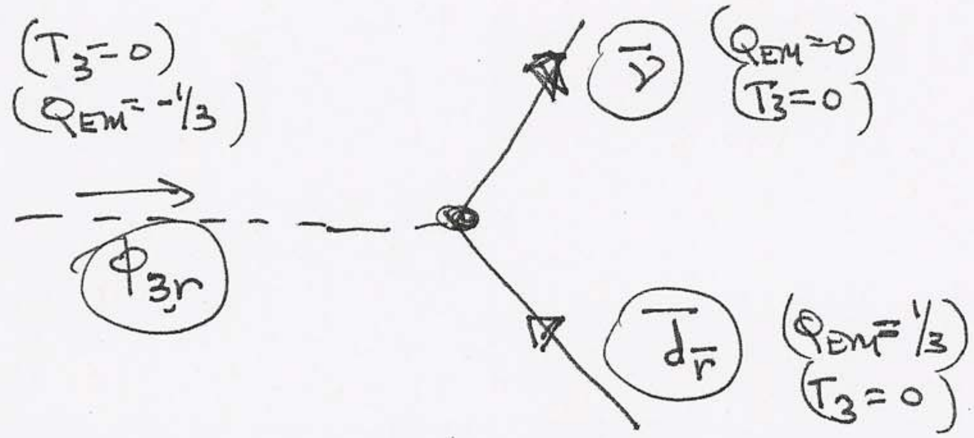
a new, "colored" Higgs triplet ϕ_3, $Q_{EM} = T_3 + \frac{Y}{2} = -1/3$

[same quantum #'s as a RH down quark d_R!]

This also mediates new interactions because of the Higgs Yukawa couplings

e.g., $\mathcal{L}_{SM} = y_d \overline{Q}_L \phi\, d_R \longrightarrow \mathcal{L}_{SU(5)} = y\, \mathbb{10} \cdot \underbrace{5 \cdot 5}_{\text{contains a singlet ✓}}$

e.g.,



$$\boxed{\Delta B = 1/3}$$