1. Take a gaussian wave packet in air, of the form

$$A(y;z) = \int d\omega\, d\mu \exp(ik_y y + ik_z z - i\omega t) \exp\left[-(\omega - \omega_0)^2/2a^2\right] \exp\left[-(\mu - \mu_0)^2/2b^2\right]$$

with $k_y = (\omega/c)\sin\mu$ and $k_z = (\omega/c)\cos\mu$: The integration can be performed after we approximate

$$\begin{aligned} k_y &= (\omega_0/c)\sin\mu_0 + (\omega_0/c)(\mu - \mu_0)\cos\mu_0 + ((\omega - \omega_0)/c)\sin\mu_0 \\ k_z &= (\omega_0/c)\cos\mu_0 - (\omega_0/c)(\mu - \mu_0)\sin\mu_0 + ((\omega - \omega_0)/c)\cos\mu_0 \end{aligned}$$

We pull out the factor $\exp(i(\omega_0/c)(y\sin\mu_0 + z\cos\mu_0) - i\omega_0 t)$ and we are left with the product of two integrals. One is

$$\int d\omega \exp(i((\omega - \omega_0)/c)(y\sin\mu_0 + z\cos\mu_0) - i(\omega - \omega_0)t) \exp\left[-(\omega - \omega_0)^2/2a^2\right];$$

which evauates to $\sqrt{2\pi a^2} \exp\left[-(a^2/2)\left[t - (y\sin\mu_0 + z\cos\mu_0)/c\right]^2\right]$: The other integral is

$$\int d\omega \exp(i((\mu - \mu_0)(\omega_0/c)(y\cos\mu_0 - z\sin\mu_0)) \exp\left[-(\mu - \mu_0)^2/2b^2\right];$$

and it evaluates to $\sqrt{2\pi b^2} \exp\left[-(b^2/2)(\omega_0/c)^2(y\cos\mu_0 - z\sin\mu_0)^2\right]$: The product is

$$2\pi ab \exp\left[-(a^2/2c^2)\left[ct - (y\sin\mu_0 + z\cos\mu_0)\right]^2\right] \exp\left[-(b^2/2)(\omega_0/c)^2(y\cos\mu_0 - z\sin\mu_0)^2\right]$$

and represents a wave packet that propagates with speed c at an angle μ_0 with respect to the z axis. The packet does not spread.

2. Take the analogous gaussian wave packet in a medium of index n

$$A(y;z) = \int d\omega\, d\mu \exp(ik_y y + ik_z z - i\omega t) \exp\left[-(\omega - \omega_0)^2/2a^2\right] \exp\left[-(\mu - \mu_0)^2/2b^2\right]$$

with $k_y = (n\omega/c)\sin\mu$ and $k_z = (n\omega/c)\cos\mu$: The integration can be performed after we approximate

$$\begin{aligned} k_y &= (n_0\omega_0/c)\sin\mu_0 + (n_0\omega_0/c)(\mu - \mu_0)\cos\mu_0 + ((\omega - \omega_0)/v_0)\sin\mu_0 \\ k_z &= (n_0\omega_0/c)\cos\mu_0 - (n_0\omega_0/c)(\mu - \mu_0)\sin\mu_0 + ((\omega - \omega_0)/v_0)\cos\mu_0; \end{aligned}$$

where v_0 is the group velocity

$$v = \frac{c}{d(n\omega)/d\omega}$$

evaluated at $\omega = \omega_0$:

We pull out the factor $\exp(i(n_0\omega_0/c)(y\sin\mu_0 + z\cos\mu_0) - i\omega_0 t)$ and we are left with the product of two integrals. One is

$$\int d\omega \exp(i((\omega - \omega_0)/v_0)(y\sin\mu_0 + z\cos\mu_0) - i(\omega - \omega_0)t)\exp\left[-(\omega - \omega_0)^2/2a^2\right];$$

which evauates to $\sqrt{2\pi a^2}\exp\left[-(a^2/2)[t - (y\sin\mu_0 + z\cos\mu_0)/v_0]^2\right]$: The other integral is

$$\int d\omega \exp(i((\mu - \mu_0)(n_0\omega_0/c)(y\cos\mu_0 - z\sin\mu_0))\exp\left[-(\mu - \mu_0)^2/2b^2\right];$$

and it evaluates to $\sqrt{2\pi b^2}\exp\left[-(b^2/2)(n_0\omega_0/c)^2(y\cos\mu_0 - z\sin\mu_0)^2\right]$: The product is

$$\tilde{A} = 2\pi ab\exp\left[-(a^2/2v_0^2)[v_0 t - (y\sin\mu_0 + z\cos\mu_0)]^2\right]\exp\left[-(b^2/2)(n_0\omega_0/c)^2(y\cos\mu_0 - z\sin\mu_0)^2\right]$$

and represents a wave packet that propagates with velocity v_0 at an angle μ_0 with respect to the z axis. We can say this because

$$\tilde{A}(t; y + v_0 t\sin\mu_0; z + v_0 t\cos\mu_0) = \tilde{A}(0; y; z)$$

In this approximation the packet does not spread, even when the medium is dispersive.

3. Take the wave packet 1 and let it pass through the air-medium interface perpendicular to the z axis (so that μ_0 is tghe average angle of incidence). The wave packet propagating in the medium is described by

$$A(y;z) = \int d\omega\, d\mu\, T(\omega;\mu)\exp(ik_y y + ik_z z - i\omega t)\exp\left[-(\omega - \omega_0)^2/2a^2\right]\exp\left[-(\mu - \mu_0)^2/2b^2\right]$$

with $k_y = (\omega/c)\sin\mu$ and $k_z = (\omega/c)\sqrt{n^2 - \sin^2\mu}$: $T(\omega;\mu)$ is the transmission coefficient. The integration can be performed if we assume that T varies slowly and we approximate

$$k_y = (\omega_0/c)\sin\mu_0 + (\omega_0/c)(\mu - \mu_0)\cos\mu_0 + ((\omega - \omega_0)/c)\sin\mu_0$$

$$k_z = (\omega_0/c)\sqrt{n_0^2 - \sin^2\mu_0} - \frac{\omega_0}{c\sqrt{n_0^2 - \sin^2\mu_0}}(\mu - \mu_0)\sin\mu_0\cos\mu_0 + (\omega - \omega_0)\frac{n_0 d(n\omega/d\omega)_0 - \sin^2\mu_0}{c\sqrt{n_0^2 - \sin^2\mu_0}};$$

We have written the last term in k_z in a way that makes it clear how it depends on the group velocity

$$v = \frac{c}{d(n\omega)/d\omega}$$

evaluated at $\omega = \omega_0$: It is also convenient to introduce the Snell angle μ_S by $\sin\mu_0 = n_0\sin\mu_S$, so that

$$\sqrt{n_0^2 - \sin^2\mu_0} = n_0\cos\mu_S$$

We can then write

$$k_y = (n_0\omega_0/c)\sin\mu_S + (\omega_0/c)(\mu - \mu_0)\cos\mu_0 + ((\omega - \omega_0)/c)\sin\mu_0$$

$$k_z = (n_0\omega_0/c)\cos\mu_S - \frac{\omega_0}{c\cos\mu_S}(\mu - \mu_0)\sin\mu_S\cos\mu_0 + \frac{(\omega - \omega_0)}{\cos\mu_S}\left(\frac{1}{v_0} - \frac{\sin\mu_S\sin\mu_0}{c}\right);$$

We pull out the factor $T(\omega_0;\mu_0)\exp(i(n_0\omega_0/c)(y\sin\mu_S + z\cos\mu_S) - i\omega_0 t)$ and we are left with the product of two integrals. One is

$$\int d\omega \exp\left[i\frac{(\omega - \omega_0)z}{\cos\mu_S}\left(\frac{1}{v_0} - \frac{\sin\mu_S\sin\mu_0}{c}\right) + i\frac{(\omega - \omega_0)y}{c}\sin\mu_0 - i(\omega - \omega_0)t\right]\exp\left[-(\omega - \omega_0)^2/2a^2\right];$$

which evaluates to

$$\sqrt{2\pi a^2}\exp\left[-(a^2/2)\left(t - \frac{z}{\cos\mu_S}\left(\frac{1}{v_0} - \frac{\sin\mu_S\sin\mu_0}{c}\right) - \frac{y}{c}\sin\mu_0\right)^2\right]$$

: The other integral is

$$\int d\omega \exp\left[i((\mu - \mu_0)(n_0\omega_0/c)\frac{\cos\mu_0}{\cos\mu_S}(y\cos\mu_S - z\sin\mu_S)\right]\exp\left[-(\mu - \mu_0)^2/2b^2\right];$$

and it evaluates to

$$\sqrt{2\pi b^2}\exp\left[-(b^2/2)(n_0\omega_0/c)^2\frac{\cos^2\mu_0}{\cos^2\mu_S}(y\cos\mu_S - z\sin\mu_S)^2\right]$$

The product is

$$2\pi ab\exp\left[-(a^2/2v_0^2)[v_0 t - (y\sin\mu_0 + z\cos\mu_0)]^2\right]\exp\left[-(b^2/2)(n_0\omega_0/c)^2(y\cos\mu_0 - z\sin\mu_0)^2\right]$$

and represents a wave packet that propagates with velocity v_0 at an angle μ_S with respect to the z axis. We can say this because

$$\tilde{A}(t; y + v_0 t\sin\mu_S; z + v_0 t\cos\mu_S) = \tilde{A}(0; y; z)$$

In this approximation the packet does not spread, even when the medium is dispersive.

4. It seems magical that the group velocity comes out to be $v_0\hat{k}$. To see that it is right and no mistakes were made, check the results for a non-dispersive medium. Then $v_0 = c/n_0$ and we can rewrite the expression

$$E = \frac{z}{\cos\mu_S}\left(\frac{1}{v_0} - \frac{\sin\mu_S\sin\mu_0}{c}\right) + \frac{y}{c}\sin\mu_0$$

in the form

$$\frac{z}{c\cos\mu_S}\left[n - n\sin^2\mu_S\right] + \frac{y}{c}n_0\sin\mu_S = \frac{n_0}{c}(z\cos\mu_S + y\sin\mu_S):$$

This suggests that in teh general case we use

$$\frac{1}{v_0} = \frac{1}{c}\left(n_0 + \omega_0 \left(\frac{dn}{d\omega}\right)_0\right),$$

andgo through th esame steps to obtain

$$E = \frac{n_0}{c}(z\cos\mu_S + y\sin\mu_S) + \frac{\omega_0 z}{c\cos\mu_S}\left(\frac{dn}{d\omega}\right)_0$$

Sustituting $y = v_0 t\sin\mu_S$ and $z = v_0 t\cos\mu_S$ one obtains, amazingly but not unexpectedly) $E = t$.

ake the wave packet 1 and let it pass through the air-medium interface perpendicular to the z axis (so that μ_0 is tghe average angle of