Radiation and scattering: why is the sky blue? As one application of the Larmor radiation formula we will study the excitation of an atom by an incident plane EM wave. The interaction of the electric field of the incident plane wave with the electrons and the nucleus causes them to oscillate at the same frequency as the wave, resulting in an oscillating electric dipole (the magnetic forces are entirely negligible except for very intense incident waves.) The accelerations of the electrons are large compared to the accelerations of the nucleus, due to the large mass ratio. The oscillating dipole then causes the atom to radiate electric dipole radiation, as discussed in class.

(a) Consider a simple model in which one electron is bound harmonically to the nucleus (i.e., the electron is subject to a restoring force, \(-m\omega_0^2z\), with \(m\) the electron mass, \(\omega_0\) a natural atomic frequency, and \(z\) the displacement from the origin, taken to be the equilibrium position of the electron). The electric field of the incident plane wave at the atom has the form \(E_z(t) = E_0 \cos(\omega t)\) (we can manage well without complex notation in this problem). Assuming that \(\omega \ll \omega_0\), show that the electron moves in phase with the electric field with an acceleration

\[
\frac{d^2z}{dt^2} = \frac{eE_0\omega^2}{m\omega_0^2} \cos(\omega t).
\]

(b) Use the Larmor radiation formula to show that the time-averaged power radiated by the charge is

\[
P_{av} = \frac{1}{4\pi\varepsilon_0} \frac{e^4E_0^2}{3m^2c^3} \left(\frac{\omega}{\omega_0}\right)^4.
\]  

Express this in terms of the wavelength \(\lambda\) of the incident wave to show that \(P_{av} \propto \lambda^{-4}\), which is a famous law derived by Lord Rayleigh. This results shows that short wavelength radiation is scattered more effectively by atoms than long wavelength radiation. This result is valid for \(\lambda \gg a\), with \(a\) the characteristic size of the atom.

(c) Use the above result to explain (i) why the sky is blue, (ii) why sunsets are red, and (iii) why it is easier to get sunburned at midday.

(d) Finally, think a bit about the polarization of the scattered radiation. Suppose you take some Polaroid sunglasses and look northward as the sun sets in the west. If you rotate the sunglasses you will notice marked intensity variations (try it). Why? Please be as specific as possible. Figure 4.3(c) in *Melissinos* may help. See also the lecture notes.

(a) We start with the very simple picture of an atom in the gas as an electron bound to a much heavier nucleus by a spring — this is what is meant by “bound harmonically”. The equilibrium position of the spring has the electron directly sitting on the nucleus. (This is a very naive model; more accurately the electron makes a transition between orbits and the “position” corresponds to the center of the orbit, which is stretched by the applied
field.) But when the electron is at other positions we have the negatively charged electron separated from the positively charged nucleus by some distance, that is, we have an electric dipole ($p = qd$ where $d$ is the distance separating the two charges $\pm q$ and the dipole points along that length from the negative to the positive charge). So if we were to apply a constant field $E_z$ to the simple harmonic atom, we'd get a displacement $z = -eE/k$, where $k$ is the spring constant, and the resulting dipole would be $p = \hat{z}e^2E/k$. Next we consider the situation in which the applied field is sinusoidal as it would be if it is the electric field associated with electromagnetic radiation (i.e. light).

$$F = ma$$

$$-eE_0 \cos(\omega t) - k z = \frac{m d^2 z}{dt^2},$$

where we neglect damping. (Melissinos includes it.) The steady-state solution to this equation is

$$z = \frac{-eE_0 \cos(\omega t)}{m(k/m - \omega^2)}.$$ 

Compare this to the solution of the LRC circuit in Assignment 8. The natural frequency is given by $\omega_0^2 = k/m$; in the limit $\omega \ll \omega_0$ we have

$$z = \frac{-eE_0 \cos(\omega t)}{m\omega_0^2}.$$ 

And the corresponding acceleration is

$$\frac{d^2 z}{dt^2} = \frac{eE_0 \omega^2 \cos(\omega t)}{m\omega_0^2}.$$ 

(b) Recall that accelerating charges radiate. The Larmor radiation formula gives the power radiated by an accelerating charge as

$$P = \frac{2}{3} \frac{e^2}{4\pi\varepsilon_0} \frac{\dot{v}^2}{c^3},$$

(Melissinos’ eq. (4.13)). Substituting the expression for acceleration found above, we get

$$P = \frac{e^4 E_0^2 \omega^4 \cos^2(\omega t)}{6\pi\varepsilon_0 m^2 \omega_0^2 c^3},$$

finally expressing this result in terms of wavelength ($\omega = c/\lambda$) results in

$$\langle P \rangle_t = \frac{e^4 E_0^2 c}{12\pi\varepsilon_0 m^2 \omega_0^2 \lambda^4},$$

where $\langle \rangle_t$ stands for the time average. Note that this is the formula found by Lord Rayleigh.

(c) (i) The above result on radiation due to “scattering” implies that blue light is scattered more than red light. Thus the sky is blue because the light of the sky is scattered light (we are not looking directly toward the source of light). Violet light should be scattered even more; so why isn’t the sky violet?

(ii) When viewing sunsets we look more directly at the source of radiation. The blue light is scattered away, leaving the red light.

(iii) In the simplest scenario we are concerned with direct, ultraviolet light, because
this light causes the greatest damage to the human skin. (So we will not worry about the scattered or reflected light, and we will not worry about the relative amounts of other frequencies in the direct light.) When the sun is low in the sky, the direct ultraviolet light travels through more atmosphere and therefore more of it is scattered.

(d) Light is a transverse wave which means that as it travels, say in the $y$ direction of the figure above, the electric field lies in the $xz$ plane. Therefore, the dipoles excited in the atmosphere also lie in the $xz$ plane. An oscillating dipole does not radiate along its axis (see fig. 4.3 in Melissinos), so dipoles oscillating in the $z$ direction do not send light to the eye. So only the component of the dipole in the $x$ direction sends radiation to the eye (along the $z$ direction). If the viewer orients his or her polaroid sunglasses to admit radiation with the electric field pointing in the $x$ direction, there is scattered light with that polarization. On the other hand, if he or she orients the sunglasses in the $y$ direction, the amount of light should be noticeably reduced.