Assignment 11 - Problem 2

Solar temperatures. Here we will try to estimate some relevant temperatures in the sun. Assume you start with an initially diffuse cloud of hydrogen and helium atoms (initially at rest), which subsequently collapse under its gravitational attraction.

(a) Use dimensional analysis to show that the total energy released in the gravitational collapse is about

\[ E_\odot = \frac{GM^2}{R} \]  \hspace{1cm} (0.1)

Calculate this energy.

(b) How does this energy compare to the total energy that could be released by (hypothetical) chemical reactions in the sun? To the total energy that could be released by converting all the hydrogen into helium? (You can assume that the cloud has the primitive abundance of helium - it does not make much difference as long as you have mostly hydrogen).

(c) Assuming that all of the energy (0.1) is converted into heat (a rather dubious assumption), estimate the interior temperature of the sun. Compare with the accepted value of the temperature at the center of the sun (look it up).

(d) The intensity of solar radiation has a peak at a wavelength 490 nm. What is the surface temperature of the sun? As you go from the surface of the sun toward the center, what is the approximate temperature gradient \(dT/dx\)? How can this temperature gradient be maintained?

(a) The energy it takes to bring two point masses \(m_1\) and \(m_2\) from infinitely far away to within a distance \(r\) is

\[ E = -G \frac{m_1 m_2}{r}, \]

where \(G = 6.6726 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2\), this is the gravitational potential energy. Dimensional analysis suggests that the gravitational energy associated with collapsing a diffuse cloud to an object the size and mass of the sun is

\[ E = -G \frac{M^2}{R}, \]
(The negative sign indicates that it does not require energy, instead energy is gained.) For the sun which has a mass \( M_\odot = 1.99 \times 10^{30} \) kg and radius \( R_\odot = 6.96 \times 10^8 \) m, we get

\[
E = -3.8 \times 10^{41} \text{ Joules.}
\]

**A digression.** If we assume the density is constant, we can calculate the numerical coefficient. Let us take \( m_1 \) to be a sphere of radius \( r \) and mass density \( \rho \) (it looks like a point mass so long as we stay outside its radius), \( m_1 \) is

\[
m_1 = \rho \frac{4\pi r^3}{3}.
\]

Let us take \( m_2 \) to be a spherical shell which after it is brought in from infinity has a radius \( r \), a thickness \( dr \) and the same mass density \( \rho \), then \( m_2 \) is

\[
m_2 = \rho 4\pi r^2 dr.
\]

The change in energy required to bring in the shell is thus

\[
dE = -G \frac{\rho 4\pi r^3}{3} \frac{\rho 4\pi r^2 dr}{r}.
\]

So the energy to make a sphere of radius \( R \) is

\[
E(R) = -\frac{16\pi^2 G \rho^2}{3} \int_0^R r^4 dr = -\frac{16\pi^2 G \rho^2}{15} R^5.
\]

Now expressing \( \rho \) in terms of the total mass \( M \)

\[
\rho = \frac{M}{\frac{4\pi R^3}{3}}
\]

and substituting we find

\[
E(R) = -\frac{3GM^2}{5R} = -2.28 \times 10^{41} \text{ Joules,}
\]

if the density were constant.

(b) The proton-proton cycle essentially turns four protons and two electrons into an alpha particle and gives off 26.7 MeV in the process. Assuming the sun
starts off entirely as protons and electrons and is converted entirely into helium (alpha particles), then the total energy would be

\[
\frac{1.99 \times 10^{30} \text{ kg}}{1.67 \times 10^{-27} \text{ kg}} \times \frac{2.67 \times 10^7 \text{ eV}}{4} \times \frac{1.60 \times 10^{-19} \text{ Joules}}{\text{eV}} = 1.3 \times 10^{45} \text{ Joules},
\]

which is the number of protons divided by four multiplied by the energy released in a proton-proton cycle. A hypothetical chemical reaction would have an energy \(10^5\) to \(10^6\) times smaller. Thus, the hypothetical chemical energy is smaller than the gravitational energy which is smaller than the fusion energy.

(c) Assuming all of the energy calculated in part (a) is converted into heat (thermal kinetic energy), we get

\[
\langle K.E. \rangle_{\text{total}} = -E_{\text{dim}}.
\]

(In a similar problem given by Kittel and Kroemer, one is told to assume that half of the gravitational potential energy is converted into thermal kinetic energy in accordance with the \textit{virial theorem}, but we are only concerned with order of magnitude here.) Let us suppose this is the average kinetic energy everywhere in the entire sun. Then the average kinetic energy per particle

\[
\langle K.E. \rangle_{\text{particle}} = \frac{\langle K.E. \rangle_{\text{total}}}{N},
\]

where \(N\) is the total number of particles. Again assuming the sun is composed purely of hydrogen, let \(N\) be the number of hydrogen atoms

\[
N = \frac{M_\odot}{m_p} = \frac{1.99 \times 10^{30} \text{ kg}}{1.672 \times 10^{-27} \text{ kg}} = 1.19 \times 10^{57},
\]

so then

\[
\langle K.E. \rangle_{\text{particle}} = \frac{3.8 \times 10^{41} \text{ Joules}}{1.19 \times 10^{57}} = 3.2 \times 10^{-16} \text{ Joules}.
\]

If we take the equipartition theorem for a monatomic gas then

\[
\langle K.E. \rangle_{\text{particle}} = \frac{3}{2} k_B T,
\]

(where we have assumed the temperature is the same throughout, which is not consistent with what we find later). But anyway this leads to

\[
T = \frac{23.2 \times 10^{-16} \text{ Joules}}{31.38 \times 10^{-23} \text{ J/K}} = 1.5 \times 10^7 \text{ K}.
\]
Melissinos (p. 193) lists the temperature at the interior of the sun as $T^\text{int}_\odot \sim 1.6 \times 10^7$ K.

(d) The electromagnetic radiation from the sun is determined by its surface temperature assuming it radiate as a black-body. Wien’s displacement law says that $\lambda_m T$ is constant, where $\lambda_m$ is the wavelength at which the blackbody radiation intensity is maximal and $T$ is the temperature of the blackbody. More specifically,

$$\lambda_m T = 2.898 \times 10^{-3} \text{ m} \cdot \text{K} = \frac{hc}{4.96511 k_B},$$

where 4.96511 is the root of the equation $e^x (5 - x) = 5$. If $\lambda_m = 490$ nm then $T^\text{sur}_\odot \approx 5900$°K.

So we go from a temperature of $T^\text{int}_\odot = 1.6 \times 10^7$ K at the center to $T^\text{sur}_\odot = 5900$ K at the surface. If the gradient is constant, it is

$$\frac{dT}{dx} = \frac{T^\text{int}_\odot - T^\text{sur}_\odot}{R_\odot} = 0.023 \text{ K/m}$$

This temperature gradient is maintained and further gravitational collapse is prevented by the radiation pressure exerted by the energy issuing from the fusion processes in the interior.