Physics 751 Final Exam
December 2004

This exam is to be pledged: write the pledge at the top of your first sheet, and sign it.

Do four questions, and make clear which question you do not want counted if you attempted five. Otherwise, I will just grade the first four.

Possibly Useful Info:
For the one-dimensional oscillator the operator $a$:

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left( x + i \frac{p}{m\omega} \right), \quad a^\dagger | n \rangle = \sqrt{n+1} | n+1 \rangle$$

Angular Momentum Operators:

$$[J_z, J_y] = i\hbar \varepsilon_{y\omega} J_z, \quad J_z^2 | jm \rangle = \hbar^2 j(j+1) | jm \rangle$$

$$J_z = J_x \pm iJ_y, \quad J_z | jm \rangle = \hbar \sqrt{j(j+1) - m(m \pm 1)} | j, m \pm 1 \rangle$$

$$Y^0_0 = (4\pi)^{-1/2}, \quad Y^\pm_1 = \mp (3/8\pi)^{1/2} \sin \theta e^{\pm i\phi}, \quad Y^0_1 = (3/4\pi)^{1/2} \cos \theta$$

$$Y_{1/2}^{\pm 1} = (15/32\pi)^{1/2} \sin \theta e^{\pm 2i\phi}, \quad Y_{1/2}^{\pm 1} = (15/8\pi)^{1/2} \sin \theta \cos \theta e^{\pm i\phi}, \quad Y^0_2 = (5/16\pi)^{1/2} (3\cos^2 \theta - 1)$$

$$\vec{\sigma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

1. A particle of mass $m$ moves in one dimension in the following potential:

$$V(x) = \infty \text{ for } x < 0, \quad V(x) = V_0 \text{ for } 0 \leq x < a, \quad V(x) = 0 \text{ for } a < x$$

PLUS a term $-\lambda \delta(x-a)$.

(a) For the case $V_0 = 0$, find the minimum value of $\lambda$, $\lambda = \lambda_0$ for which a bound state is possible.

(b) Suppose now $V_0$ is nonzero but small. Find an approximate expression for how this changes the value of $\lambda_0$ found above, for $V_0$ positive.

(c) Sketch the zero-energy wave functions for small $V_0$, for the cases of $V_0$ both positive and negative, and with $\lambda_0$ adjusted to give a zero-energy bound state.
(d) If \( V_0 \) is negative, increasing its strength will at some point create a second bound state. Sketch the wave function of that bound state when it has extremely small binding energy.

2. (a) Assuming \( p, x \) obey canonical commutation relations, find the commutation relations for the simple harmonic oscillator raising and lowering operators (see formulas at beginning of exam). Generalize your result to find \([a, (a^\dagger)^n]\). (First take \( n = 2 \), then \( n = 3 \), look for the pattern, and find the general result by induction.)

(b) Prove that \( e^{a^\dagger} |0\rangle \) is an eigenstate of the (non-Hermitian) annihilation operator \( a \). Find its eigenvalue, and normalize it.

(c) Denote the normalized eigenstate in (b) by \( |\lambda\rangle \). Represent \( |\lambda\rangle \) as an infinite series in the simple harmonic oscillator energy eigenstates \( |n\rangle \), and prove that integrating \( |\lambda\rangle\langle\lambda| \) over all possible values of \( \lambda \), gives the identity operator (suitable normalized). (Hint: write \( \lambda = re^{i\theta} \) and integrate over the complex plane.)

3. An electron in a three-dimensional harmonic oscillator potential has the wave function

\[
\psi(x, y, z) = A(x/a)^2 e^{-(x/a)^2}
\]

(a) Find the possible results of measuring \( J^2, \mathbf{J}_z \).

(b) Find the relative probabilities for the different possible values you found in (a).

4. (a) For a system consisting of two spins one, what are the possible values of the total angular momentum?

(b) Writing the base kets of such a system in the notation \(|1, m\rangle \otimes |1, m'\rangle\), find the correctly normalized ket corresponding to total angular momentum zero.

5. A spin-1/2 system is in the eigenstate of the operator \( S_x + S_z \) with the largest possible eigenvalue. It is rotated about the \( y \)-axis by \( \pi/2 \), then about the \( x \)-axis by \( \pi/2 \). What is the probability that in the final state the component of spin in the \( z \)-direction will be \( +\frac{1}{2} h \)?