1. (a) Defining the delta function as the limit of a narrow Gaussian wave packet (see the web notes on Fourier Series, etc.) prove it has the following properties:

\[ \int \delta(x) \, dx = 1, \quad \delta(x) = 0 \text{ for } x \neq 0. \]

\[ \delta(x) = \delta(-x), \quad \delta(ax) = \frac{1}{|a|} \delta(x), \]

\[ \int \delta(a-x)\delta(x-b) \, dx = \delta(a-b). \]

(b) Suppose you define the delta function by:

\[ \Delta(x) = \lim_{L \to 0} \Delta_L(x), \quad \text{where } \Delta_L(x) = 0 \text{ for } |x| \geq L/2, \quad \Delta_L(x) = 1/L \text{ for } |x| < L/2. \]

Does this function have all the above properties?

2. Use Mathematica, Maple or Integral Tables to find the integral of \((\sin x)/x\) from 0 to \(\pi\) and from 0 to infinity. Use your result to estimate the overshoot that appears in a Fourier series representation of a step function (Gibbs’ phenomenon).

3. Suppose at \(t = 0\), a free particle of mass \(m\), in one dimension, has a Gaussian wavefunction

\[ \psi(x, t = 0) = \frac{1}{(\pi \Delta^2)^{1/4}} e^{-x^2/2\Delta^2}. \]

By taking a Fourier transform and putting in explicit time-dependence, find the form of the wavefunction as a function of time, and provide a physical interpretation in terms of finding the particle somewhere.

4. Denoting the lowest energy eigenstate in an infinite square well by \(\psi_0\), and the first excited state by \(\psi_1\), describe the behavior of the probability distribution as a function of time for the state \(\psi_0 + \psi_1\) (appropriately normalized). Find the expectation value of the probability current at the midpoint of the well as a function of time. How would your analysis be different for the state \(\psi_0 + i\psi_1\)?

5. Prove Parseval’s Theorem:
If \( f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} a(k) e^{ikx} \), then \( \int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} \frac{dk}{2\pi} |a(k)|^2 \).

6. Prove the rule for the Fourier Transform of a convolution of two functions:

If \( f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} a(k) e^{ikx} \), \( g(x) = \int_{-\infty}^{\infty} \frac{dk'}{2\pi} b(k') e^{ik'x} \),

then \( \int_{-\infty}^{\infty} f(x-x') g(x') \, dx' = \int_{-\infty}^{\infty} \frac{dk}{2\pi} a(k) b(k) e^{ikx} \).