Physics 752 Final Exam
May 2, 2003

1. (a) Using a Gaussian trial wave function, \( \psi(x) = Ce^{-ax^2} \) find an approximate ground state energy for a particle of mass \( m \) in a V-shaped attractive potential, \( V(x) = V_0|x| \).

(b) Describe how you would find an approximate energy for the first excited state by modifying the Gaussian wavefunction appropriately (you don’t have to carry it through).

2. A one-dimensional simple harmonic oscillator is placed in a uniform external field,

\[
H = H^0 + H^1 = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 - qfx.
\]

Using perturbation theory in \( H^1 \), and denoting the \( n \)th state by \( |n> \), find:

(a) the first-order shift in energy for the \( n \)th state,

(b) the first-order change in the wavefunction of the \( n \)th state (in terms of contributions from other states),

(c) the second-order shift in the energy of the \( n \)th state.

(d) Check your results by solving the problem exactly.

3. (a) Write down Fermi’s Golden Rule for a quantum transition rate.

(b) For a photon emitted (in an atomic transition) into a solid angle \( d\Omega \), show that the density of states as a function of energy is \( \frac{V \omega^2 d\Omega}{(2\pi)^3 \hbar c^3} \), with normalization volume \( V \).

(c) Using the quantized form of the vector potential,

\[
A(x,t) = (1/\sqrt{V}) \sum_k \sum_\alpha e^{i(k_x x - \omega t)} [a_{k,\alpha}^* (t)e^{i\epsilon_\alpha} + a_{k,\alpha} (t)e^{-i\epsilon_\alpha}]
\]

write down an expression for the spontaneous transition rate \( 2p \) to \( 1s \) in hydrogen, using the \( A.p \) form for the interaction.

(d) Justify (briefly) dropping the \( e^{i k_x x} \) term.
(e) Transform your matrix element so that it is a function of the operator \( x \) rather than \( p \).

(f) What is the order of magnitude of the expectation value of \( |x_{BA}|^2 \) in this transition in terms of the Bohr radius and the electron charge?

(g) Write down a complete expression for the spontaneous transition rate in terms of \( |x_{BA}|^2 \). If you can’t do the angle integrals, make a guess.

(h) How is the spontaneous transition rate related to the inverse transition rate for a hydrogen atom in the ground state in a macroscopic electromagnetic field of the appropriate frequency?

4. For scattering of a plane wave from a potential, the wave function is taken to be:

\[
\psi_k = e^{ikr} + \psi_{sc}, \quad \text{where } \psi_{sc} \to f(\theta, \phi, k) \frac{e^{ish}}{r} \text{ at large } r.
\]

If \( V(r) = V(r) \), there is no \( \phi \) dependence, and the partial wave amplitudes \( a_l(k) \) are defined by:

\[
f(\theta, k) = \sum_l (2l+1)a_l(k)P_l.
\]

The incident plane wave can be written:

\[
e^{ikz} = e^{ikr \cos \theta} = \sum_l i^l (2l+1) j_l(kr) P_l(\cos \theta)
\]

where for large \( r \)

\[
j_l(kr) \to \frac{\sin(kr - l\pi/2)}{kr}.
\]

In the presence of a (finite range) potential, this asymptotic dependence becomes:

\[
R_l(r) \to \frac{A_l \sin[kr - l\pi/2 + \delta_l(k)]}{r}
\]

(a) From the formulas above, derive an expression for \( a_l(k) \) in terms of \( k \) and \( \delta_l \).

(b) What is the total cross-section in terms of \( f(\theta, k) \)? From that result, find it in terms of \( k \) and the \( \delta_l \)’s.

(c) Derive the connection between the total cross-section and the forward scattering, and give a brief physical explanation.

(d) For an arbitrary potential equal to zero beyond \( r = a \), is there a limit to the size of the \( l = 0 \) cross-section as \( k \) goes to zero? Explain your answer.