Non-abelian statistics
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Non-abelian statistics are just plain interesting.

They probably occur in the $\nu = 5/2$ FQHE, and people are constructing time-reversal-invariant models which realize them.

One conceivable application is in quantum computing.
Outline:

1. What are non-abelian statistics?
2. Turning the problem into non-intersecting loops
3. Finding lattice models
4. Finding field theories
5. What this has to do with the $S$ matrix

work with E. Fradkin
related work with E. Ardonne and E. Fradkin
In quantum mechanics, the wavefunction depends on the positions of the particles and their quantum numbers $i_1, i_2, \ldots$. To make the notation simpler, we just denote the labels $i_1, i_2, \ldots$ by a single one $a$:

$$\psi_a(x_1, x_2, \ldots)$$

The statistics of the particles comes from the behavior of $\psi$ under the interchange $x_1 \leftrightarrow x_2$.

To be precise, one adiabatically changes states, keeping the particles far from each other.
In 3+1 dimensions, the only statistics for identical particles are \textit{bosonic} and \textit{fermionic}:

\[
\psi(x_1, x_2, \ldots) = \pm \psi(x_2, x_1, \ldots)
\]

Exchanging twice is a closed loop in configuration space.

In 3+1 dimensions, this can be deformed to the identity.
In 2+1 dimensions, it cannot!

Identical particles can pick up an arbitrary phase: fractional statistics.

Such particles are called anyons.

They have been observed in the fractional quantum Hall effect. (These have fractional charge as well).
In 2+1 dimensions, the statistics comes from behavior under braiding.

The lines are the world-lines of the particles in spacetime.
Even more spectacular phenomena than anyons can occur under braiding.

Particles can change quantum numbers!

\[ \psi_a(x_1, x_2 \ldots) = \sum B_{ab} \psi_b(x_2, x_1 \ldots) \]

When the matrix \( B \) is diagonal with entries \( \pm 1 \), particles are bosons and fermions.

When \( B \) is diagonal with entries \( e^{i\alpha} \), particles are anyons.

When \( B \) is not diagonal, can get non-abelian statistics.

With non-abelian statistics, how the wavefunction changes depends on the order in which the particles are braided.
A convenient way of describing braiding is to **project** the world line of the particles onto the plane. Then the braids become **overcrossings**

\[
B = \begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{overcrossing.png}}
\end{array}
\]

and **undercrossings**

\[
B^{-1} = \begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{undercrossing.png}}
\end{array}
\]

In 2+1 dimensions, it matters which direction you braid!
To study the statistics, it is very convenient to represent the braids algebraically.

Label the particles \(1 \ldots N\). The braid generator \(B_i\) braids the \(i\)th with the \(i + 1\)th particle:

\[
\begin{align*}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{align*}
\]

\[
= B_3^{-1} B_4 B_3 B_1 B_1
\]
For non-abelian statistics, we need the $B_i$ to be matrices so, e.g., $B_i B_{i+1} \neq B_{i+1} B_i$.

Not any set of matrices $B$ corresponds to a braid. The generators of the braid group must satisfy

$$B_i B_{i+1} B_i = B_{i+1} B_i B_{i+1}$$
One simple example is well-known from knot theory. Let $e$ be a **monoid**:

\[
\begin{array}{c}
| & | \\
I & e
\end{array}
\]

Then let the braid $B_i$ be related to $e_i$ by

\[
\begin{array}{c}
\times \\
I - qe_i
\end{array}
\]

$$B_i = I - qe_i$$

for some parameter $q$. 
The $B_i$ defined this way satisfy the braid-group relation when the $e_i$ satisfy the Temperley-Lieb algebra

$$e_i^2 = de_i \quad e_i e_{i \pm 1} e_i = e_i$$

where $d = q + q^{-1}$. Note that closed loops are weighted by $d$. 
When
\[ d = 2 \cos[\pi/(k + 2)] \]
i.e. \[ q = e^{i\pi/(k+2)}, \]
these are the statistics of Wilson loops in $SU(2)_k$ Chern-Simons theory.

Witten

For $SU(2)_k$, the particles (Wilson loops) are in the “spin-1/2” representation. For $SO(3)_k$, they are in the “spin-1” representation. We can “fuse” together the spin-1/2 particles to get spin-1 particles by $I - e/d$:

\[
\begin{array}{c}
\text{---} \\
\text{---}
\end{array}
\quad = \quad
\begin{array}{c}
\text{---} \\
\text{---} \\
\text{---}
\end{array}
\quad - \frac{1}{d}
\]
The braid is then

\[ B = q^2 I - X + q^{-2} E \]

In this case, one can check that closed loops get a weight

\[ d^2 - 1 = 1 + q^2 + q^{-2} \]
The problem: find a quantum Hamiltonian acting on a two-dimensional Hilbert space which has the above properties.

One answer: Strongly-coupled Yang-Mills theory with a Chern-Simons term

Witten; Ardonne, Fendley and Fradkin

\[ S = S_{CS} + S_{SC} \]

\[ S_{CS} = \frac{k}{4\pi} \int_M e^{\mu\nu\alpha} \text{Tr} \left[ A_\mu \partial_\nu A_\alpha + \frac{2}{3} A_\mu A_\nu A_\alpha \right] \]

\[ S_{SC} = \frac{1}{2e^2} \int_M \text{Tr} \left[ F_{0i} F^{0i} \right] \]

The ground states are Wilson loops.
The excited states are Polyakov loops.
This is not completely satisfactory: we don’t know how to compute much outside of the topological limit, there is no obvious lattice model, and we don’t know if a quantum critical point separates this phase from an ordered phase.

The trick: Think of the basis elements of the Hilbert space as states in a classical 2d theory. Then find a quantum Hamiltonian which ground-state wavefunction

\[ \Psi_0(s) = \frac{e^{-\beta E_s}}{Z} \]

where the state \( s \) has energy \( E_s \), and \( Z \) is the classical 2d partition function

\[ Z = \int_s e^{-\beta E_s - \beta E_s^*} \]

Equal-time correlators in the quantum ground state are classical 2d correlators

\[ \langle \phi_1 \phi_2 \rangle = \frac{1}{Z} \int_s \phi_1 \phi_2 \ e^{-\beta E_s - \beta E_s^*} \]

Note that we need to weight configurations by \( |\Psi_0|^2 \). Rokhsar and Kivelson
Our planar projection suggests we look for a quantum loop gas, where the basis states of the two-dimensional Hilbert space are closed loops. They are closed loops because of our rules for braiding.

This is now a projection onto 2d space from 2+1d spacetime. In the $SU(2)_k$ and $SO(3)_k$ cases, we want the loops to have weights $d$ and $d^2 - 1$ respectively.

Freedman, Nayak, Shtengel, Walker and Wang; Fendley and Fradkin
The excitations with non-abelian statistics will be defects in the sea of loops. If the defects are deconfined, then they will braid with each other like the loops in the ground state.

Another name for a model with such behavior is a quantum spin liquid.
Let's make this concrete in the (abelian) quantum eight-vertex model.

The basis elements of the Hilbert space of the 2+1-dimensional model are the configurations of the eight-vertex model, e.g.
The classical model assigns Boltzmann weights to each vertex:

\[
\text{with } a = b = 1 \text{ to preserve rotational symmetry.}
\]

We want to define a 2+1-dimensional quantum system with this two-dimensional classical configurations as the basis elements of the Hilbert space. The ground-state amplitude of a 2+1-dimensional quantum state \( s \) is the same as the 2d classical Boltzmann weight:

\[
\Psi_0(s) = \frac{e^{-\beta E_s}}{Z(c^2, d^2)}
\]

where \( Z \) is the classical 2d partition function.
We construct an operator $\hat{Q}_p$ for each plaquette which annihilates the weighted sum of states. We take the Hamiltonian to be

$$H = \sum_p \hat{Q}_p^\dagger \hat{Q}_p$$

Any state $|\psi_0\rangle$ annihilated by all the $\hat{Q}_p$ is a quantum ground state with energy $E = 0$.

Any correlator in such a ground state is equal to a correlator in the classical 8-vertex model!
We can construct such a Hamiltonian by making $Q_p$ a projection operator, of the form

$$Q_p = V_p - F_p = \begin{pmatrix} v & -1 \\ -1 & v^{-1} \end{pmatrix}$$

The non-diagonal flip term $F_p$ reverses the arrows around one plaquette. This preserves the restriction that the number of arrows pointing in at any vertex be even.

The potential on each plaquette is then $V_p = c^{n_c - \tilde{n}_c} d^{m_d - \tilde{n}_d}$, where $n_c$ and $n_d$ are the number of $c$ and $d$ vertices at the corners of this plaquette, while $\tilde{n}_c$ and $\tilde{n}_d$ are the number on the flipped plaquette.
When \( a = b = c = d = 1 \) (infinite temperature in the classical model), the Hamiltonian is gauge invariant, so this model is a \( \mathbb{Z}_2 \) lattice gauge theory. Gauge-invariant operators are Wilson loops.

In fact, the ground state is described by a topological field theory.
On genus $g$, there are $2^g$ ground states; the flip preserves the number of up arrows mod 2 around a given cycle. Correlation functions of Wilson loops do not depend on distance, but only on how they intersect each other.

This special point is trivial to solve, because every plaquette is independent of the others (no potential). It was originally introduced because the anyonic excitations (in a generalized version) are valuable for error correction in quantum computers. Kitaev
The 8-vertex model in general is not free-fermionic – it is equivalent to two coupled Ising models. From Baxter’s exact results in 2d, we know that the quantum eight-vertex model contains topological order, conventional order, and conformal quantum critical lines.
In field theory:

The continuum limit of the square-lattice quantum dimer model is argued to be 2+1-dimensional scalar field theory with Hamiltonian

\[ H = \int d^2 x \left[ \Pi^2 + \kappa (\nabla^2 \phi)^2 \right]. \]

where \( \Pi = \dot{\phi} \) Henley

This theory arises in three-dimensional classical statistical mechanics as the continuum description of a Lifshitz point, where commensurate, incommensurate and disordered phases meet.

We thus dub it the quantum Lifshitz theory.
This theory is quadratic in \( \phi \), and so trivially solvable. However, there is a particularly elegant way of solving it.

The canonical commutation relations are

\[
[\phi(x, t), \Pi(y, t)] = i\delta^2(x - y).
\]

In the Schrodinger picture, \( \Pi = -i\frac{\delta}{\delta \phi} \), giving the Schrodinger equation for the wavefunctional \( \Psi[\phi] \):

\[
\int d^2x \left[ -\left(\frac{\delta}{\delta \phi}\right)^2 + \kappa(\nabla^2 \phi)^2 \right] \Psi[\phi] = E\Psi[\phi].
\]

Define

\[
Q(x) = \frac{\delta}{\delta \phi} + \kappa(\nabla^2 \phi)
\]

and the quantum Hamiltonian is

\[
H = \int d^2x \, Q^\dagger Q
\]
The ground state has $E = 0$, and the ground-state wave functional satisfies

$$Q(\vec{x})\Psi_{GS}[\phi] = 0.$$ 

for any $\vec{x}$. This equation is easy to solve, giving

$$\Psi_{GS}[\phi] = e^{-S[\phi]}$$

where

$$S = \frac{\kappa}{2} \int d^2x (\nabla \phi)^2.$$ 

This is the action of a free two-dimensional boson!
Equal-time correlators in the quantum ground state are those of a classical 2d free-boson field theory

\[ \langle e^{i\phi(\vec{x},t)} e^{i\phi(\vec{y},t)} \ldots \rangle_{GS} = \frac{\int [D\phi] e^{-2S} e^{i\phi(\vec{x})} e^{i\phi(\vec{y})} \ldots}{\int [D\phi] e^{-2S}} \]

The factor of two is because correlators in a quantum theory are weighted by \(|\Psi_0|^2\).

This 2+1-dimensional theory is not Lorentz-invariant. Space and time scale differently: the dynamical critical exponent \(z = 2\).

\(\kappa\) parametrizes a conformal quantum critical line. The wave functional is invariant under conformal transformations in space, but not in time.
To try to generalize this to the non-abelian case, let's think instead in terms of a 1+1d quantum model. The loops are the world lines of the particles of the 1+1d theory.

**The upshot:** Just think of the 2+1d world lines projected down to 1+1d.
We need to ensure that the world lines have the right braiding.

In 1+1d, particles can’t go around each other. **1+1d “braiding” is given by the $S$ matrix!**

It’s well-known from knot theory that if $S(\theta)$ obeys the Yang-Baxter equation, then

Akutsu, Deguchi and Wadati

\[
B = \lim_{\theta \to \infty} S(\theta), \quad B^{-1} = \lim_{\theta \to \infty} S(-\theta)
\]

where $\theta$ is the rapidity difference of the two particles.
The braiding of the $SO(3)_k$ Chern-Simons theory corresponds to the scattering of the $Q$-state Potts model with

$$Q = d^2 = (q + q^{-1})^2 = 4 \cos^2 \left( \frac{\pi}{k+2} \right)$$

The weight of $d^2 - 1 = Q - 1$ per loop is the number of different domain walls between the Potts spins.
All this has been to show:

The Hilbert space of the $SO(3)_k$ quantum loop gas is given by the configurations of the $Q$-state Potts field theory.

This yields a topological field theory when the weight per loop is independent of its length. This occurs at infinite temperature in the 2d classical model.

In the Ising case $Q = 2$ (weight 1 per loop), this reduces to Kitaev’s $\mathbb{Z}_2$ model.
For non-integer $Q$, the $S$ matrices describe scattering of “restricted” kinks in a potential with multiple minima.

Smirnov; Chim and Zamolodchikov; Fendley and Read

These braid matrices obey the Jones-Wenzl projector automatically.

2d classical lattice models with this $S$ matrix and domain walls with weight $4 \cos^2 \left( \frac{\pi}{k + 2} \right) - 1$ behavior are called dilute $A_k$ models.

Warnaar, Nienhuis, and Seaton
For example, in the dilute $A_3$ model, there are three spins 1, 2, 3, with the restriction that state 1 cannot be next to 3 (a RSOS model).

The Boltzmann weights are such that only regions of 1 and 2 are minima. Thus there are two kinds of domain walls: between 1 and 2, and between 2 and itself (spin 3).

Because of the restriction, the “number” of different domain walls is

$$\frac{1 + \sqrt{5}}{2} = 2 \cos \left( \frac{\pi}{5} \right) = 4 \cos^2 \left( \frac{\pi}{5} \right) - 1.$$
To determine the phase diagram, remember that a configuration \( s \) is weighted by \( |\psi(s)|^2 \) in the quantum theory.

Thus each weight is squared: each loop gets a weight \((Q - 1)^2\).

This suggests that the phase diagram is that of the \( Q_{\text{eff}} \)-state Potts model, where

\[
Q_{\text{eff}} - 1 = (Q - 1)^2 = (d^2 - 1)^2 = 1 + 2 \cos[2\pi/(k + 2)]
\]
There is a critical point when $Q_{eff} \leq 4$: $k = 1, 2, 3$. $k = 1$ is trivial, $k = 2$ is abelian. $k = 3$ is the “Lee-Yang” theory (the braiding rules are those of the Lee-Yang CFT).

The critical point with

$$Q_{eff} = 1 + \left( \left( \frac{1 + \sqrt{5}}{2} \right)^2 - 1 \right)^2 = \frac{5 + \sqrt{5}}{2}$$

is the conformal field theory with $c = \frac{14}{15}$.

$G_2$ coset !?!

This determines the equal-time correlators in the ground state of the quantum loop gas.
• There are lattice models and field theories which exhibit topological order and conformal quantum critical points. For $SO(3)_k$, Potts; for $SU(2)_k$, $O(n)$ model.

• Equal-time correlators at the critical points can be computed exactly.

• There is a gapped field theory with Chern-Simons topological field theory describing the ground state.

• The excitations of this theory obey non-abelian statistics.