Relativistic Effects in the Global Positioning System

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The Global Positioning System (GPS) provides a superb opportunity to introduce relativity concepts to undergraduate students, including non-physics majors. Familiarity with the numerous applications of GPS motivates students to understand relativity. A few fundamental principles need to be introduced, including the postulates of special relativity and the universality of free fall. Then a series of thought experiments leads to the breakdown of simultaneity, the Sagnac effect, the first-order Doppler effect, gravitational frequency shifts, and time dilation. This article presents this chain of thought and explains the essential role of special and general relativity in the GPS.

I. INTRODUCTION

This paper develops a series of thought experiments based on a few fundamental relativity principles, and discusses how the predicted effects are incorporated into the GPS. Important relativistic effects on GPS satellite clocks include gravitational frequency shifts and time dilation. These effects are so large that if not accounted for, the system would not be effective for navigation. Reference clocks on earth's geoid are similarly influenced by time dilation (due to earth's rotation) and gravitational frequency shifts, relative to clocks at infinity. The frequency differences between clocks in orbit, and reference clocks on earth's surface, are very important in the GPS. Constancy of the speed of light is essential for navigation using GPS. This principle also leads directly to the relativity of simultaneity and to the Sagnac effect, that must be accounted for when synchronizing clocks in the neighborhood of

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earth or comparing clocks that are thousands of km apart on earth's surface but that have one or more GPS satellites in view at the same time. The relativity of simultaneity, the constancy of c, and the first-order Doppler effect are intimately related.

In the next section, three important relativity principles are described. Navigation based on the constancy of the speed of light, and the Sagnac effect, are discussed in Sect. III. Sect. IV develops the relativity of simultaneity from the constancy of c, obtains the first order correction to the time component of the Lorentz transformations, and relates the Sagnac effect to the relativity of simultaneity. In Sect. V the first-order Doppler effect is derived from the time transformation, and this, together with the weak equivalence principle, is used in Sect. VI to derive the gravitational frequency shift. Time dilation is derived in Sect. VII from the constancy of the speed of light. Sect. VIII discusses some additional implications of relativity in the GPS.

II. RELATIVITY PRINCIPLES

The postulates of the special theory of relativity may be stated as follows:

1. The laws of physics have the same form in all inertial reference frames.

2. The speed of light c is a constant independent of the motion of the source.

The third principle that is required is the weak principle of equivalence, or the universality of free fall. This can be stated as:

3. Over a small region of space and time, it is impossible to distinguish between a gravitational field due to mass, and a fictitious gravitational field due to acceleration.

In the GPS, the law of inertia holds to a high degree of approximation in a freely-falling, locally inertial frame whose origin is attached to the center of the earth but which is not rotating. This is called the "Earth-Centered Inertial," or ECI frame. The significance of this is that in such a reference frame, the speed of light, $c = 299792458$ m/s, is essentially constant even though the earth is accelerated towards the sun. In the earth-centered earth-fixed rotating reference frame (the "ECEF" frame), light travels in a spiral path and clock synchronization becomes problematic because of the Sagnac effect. Willingness on the part of the student to view GPS navigation from a local inertial frame is an important step in understanding GPS.

If an elevator accelerates upwards, there is an induced gravitational field of strength

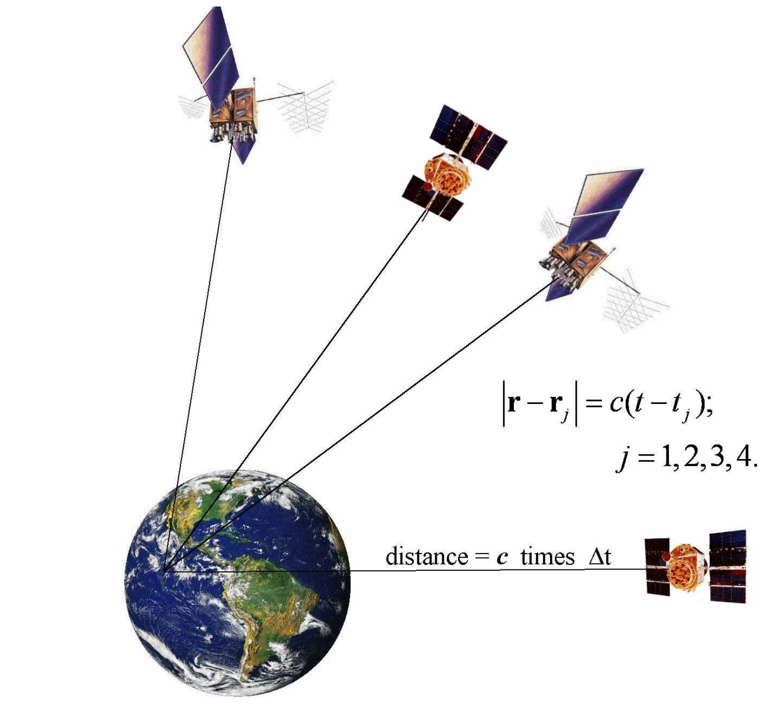

FIG. 1: Navigation in the GPS is based on constancy of the speed of electromagnetic signals.

equal to, but opposite to, the acceleration. The true gravitational field and the induced gravitational field can be superposed vectorially. If the elevator should fall freely, the induced acceleration is upwards and exactly cancels the true gravitational field so that the effective field strength in the freely falling frame vanishes. As the earth falls towards the sun, there is an induced gravitational field that is equal and opposite to the true gravitational field due to the sun so that at the center of mass of earth, the net gravitation field strength is zero. This is important in the GPS because it means that the net effect of the sun (and other solar system bodies) comes in only through tidal forces and tidal gravitational potentials. The effect on GPS clocks is due to the resulting tidal potential and is negligible. Detailed study of how this comes about shows that the relativity of simultaneity plays a crucial role in cancelling out the gravitational field strengths at earth's center due to other solar system bodies.

III. CONSTANCY OF THE SPEED OF LIGHT

A. Navigation equations

Figure 1 illustrates how navigation in the GPS is performed, based on the constancy of c. Atomic clocks in the satellites are synchronized in the underlying inertial frame (ECI) frame. Time signals from at least four satellites are received at the time and position $\{t, \mathbf{r}\}$. Messages within the signals provide information about the time and position of the transmission events, $\{t_j, \mathbf{r}_j\}$, with $j = 1, 2, 3, 4$. The four equations

$$c(t - t_j) = |\mathbf{r} - \mathbf{r}_j|; \quad j = 1, 2, 3, 4 \tag{1}$$

express constancy of the speed of light. The equations are solved by the receiver to provide position $\mathbf{r}$ and time t. This system of equations is nonlinear so the solution is nontrivial and is usually performed by linearizing the equations, assuming an approximate position (e.g., the center of earth or the last best known position) and solving by iteration. An error of 3 nanoseconds (ns) in time would correspond to a position error of about 1 meter. In practice the error in the solution for time at the receiver is typically no more than a dozen nanoseconds or so. Most hand-held receivers cannot provide time to the user with such accuracy. Time in the GPS is described as "coordinate time," time of a network of synchronized clocks that would be established using constancy of the speed of light in the ECI reference frame.

In practice navigation is more complicated, as corrections for propagation delay due to electrons in the ionosphere, and water vapor in the troposphere, may be quite important. Receivers with more than four channels are commonplace; this provides an opportunity for reduction of navigation errors by averaging. These issues have nothing to do with relativity, however.

B. Sagnac effect

The navigation equations cannot be written as they are in Eq. (1), in the ECEF (Earth-Centered, Earth-Fixed) reference frame. During propagation of the signal from transmitter to receiver, the receiver most likely moves. Even if the receiver is at rest on earth's surface, earth rotation will carry the receiver into a different position while the signal propagates to the receiver. In the rotating frame one may account for this to sufficient accuracy by writing the navigation equations as

$$t = t_j + \frac{|\mathbf{r}(t) - \mathbf{r}_j|}{c} = \frac{|\mathbf{r}(t_j) + \mathbf{v} \times (t - t_j) - \mathbf{r}_j|}{c} \tag{2}$$

where $\mathbf{r}(t)$ is the receiver position at time t and $\mathbf{v}$ is the velocity of the receiver at the time of the transmission event. Since the receiver velocity is much smaller than c, the velocity term is quite small and the equation can be solved by iteration. We define the range from transmitter to receiver at the transmission time as

$$\mathbf{R} = \mathbf{r}(t_j) - \mathbf{r}_j \,. \tag{3}$$

Neglecting the velocity term entirely,

$$t = t_j + \frac{|\mathbf{r}(t_j) - \mathbf{r}_j|}{c} = t_j + \frac{R}{c} \,. \tag{4}$$

This would give the signal's time of arrival at the receiver if the receiver were fixed in inertial space. Substituting this value of t back in to the right side of Eq. (2) and expanding to first order in v,

$$t = t_j + \frac{\sqrt{R^2 + 2\mathbf{R} \cdot \mathbf{v}(t - t_j)}}{c} \approx t_j + \frac{R}{c} + \frac{\mathbf{R} \cdot \mathbf{v}}{c^2} \,. \tag{5}$$

This is illustrated in Figure 2. If the receiver velocity is due only to earth rotation, then

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}(t_j) \,. \tag{6}$$

Then the Sagnac correction term can be rewritten as

$$\Delta t_{\mathrm{Sagnac}} = \frac{\mathbf{R} \cdot \mathbf{v}}{c^2} = \frac{2\boldsymbol{\omega} \cdot \mathbf{A}}{c^2} \tag{7}$$

where $\mathbf{A}$ is the vector area given by

$$\mathbf{A} = \frac{1}{2}\mathbf{r}(t_j) \times \mathbf{R} \,. \tag{8}$$

The area $\mathbf{A}$ is the area swept out by the tip of a vector from the rotation axis to the signal pulse as it propagates from transmitter to receiver. The dot product projects this area onto a plane parallel to the equatorial plane. The correction is very important when comparing atomic clocks at known earth-fixed locations, that have one or more GPS satellites in view at the same time. The correction can be hundreds of ns. See Figure 3 for an illustration of the "common-view" GPS geometry for comparison of atomic clocks that are separated by large distances.

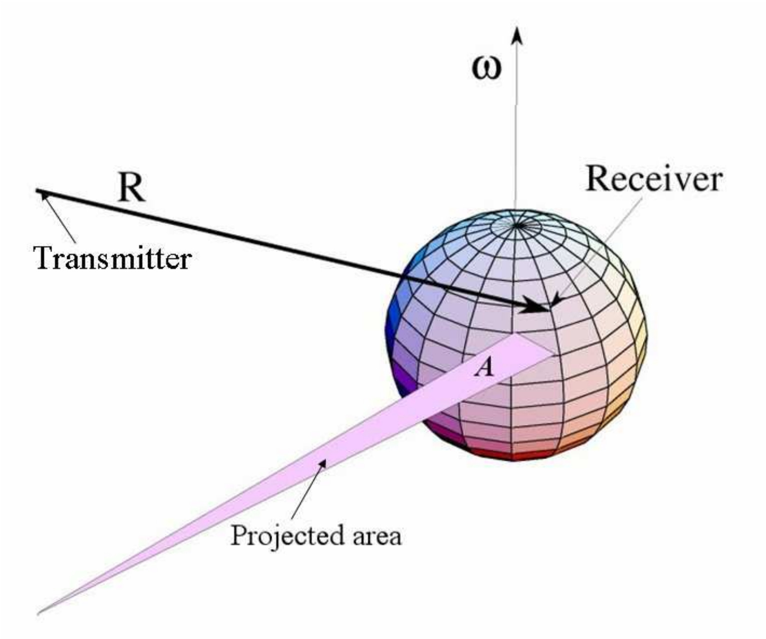

FIG. 2: The Sagnac correction is proportional to the area swept out by a vector from the rotation axis to the tip of the signal pulse, projected onto the equatorial plane.

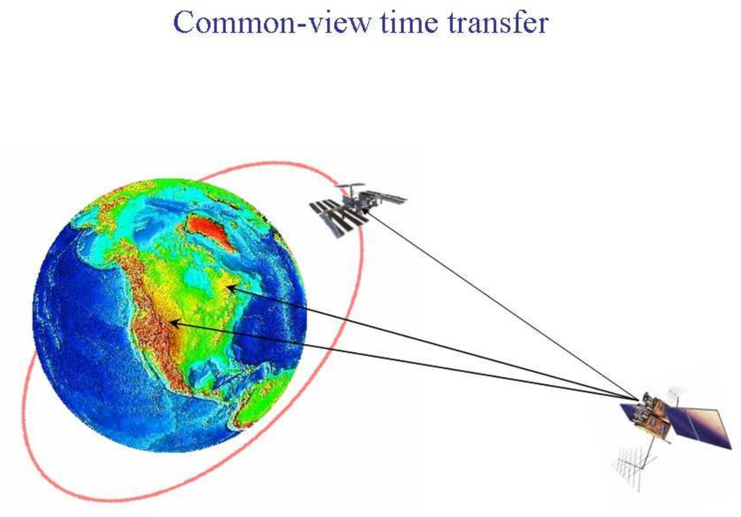

FIG. 3: In the common-view method of time comparison, the Sagnac correction accounts for receiver motion during signal propagation. Receiver positions on earth are presumed known.

IV. RELATIVITY OF SIMULTANEITY

We next consider a simple thought experiment, again based on the constancy of c, that shows how the synchronization of clock networks depends on the state of motion of the

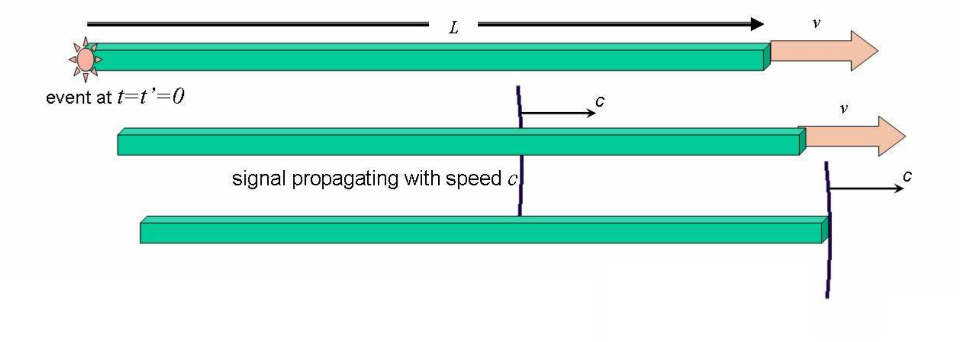

FIG. 4: Analyzing a light signal that propagates along a moving rod leads to the relativity of simultaneity.

observer. In Figure 4 we imagine a rod of length $L = x$ in motion with speed v relative to the "laboratory," and observers at rest with respect to the rod. Let a light signal originating from the left end of the rod, proceed to the right end. Calculate to first order in v the time it takes for light to travel along the rod. (Length contraction is of second order so plays no role here.) The observer in the "moving" frame, whose measurements we denote with primes, would write

$$t' = x/c\,. \tag{9}$$

On the other hand, the observer in the laboratory sees that the light wave is catching up to the front end of the rod with a relative speed $c - v$. It therefore requires a time (we do not put primes on these measurements)

$$t = \frac{x}{c-v} = \frac{x}{c}(1 - v/c)^{-1} \approx \frac{x}{c} + \frac{vx}{c^2}. \tag{10}$$

Thus

$$t' = t - \frac{vx}{c^2}\,. \tag{11}$$

This expresses the breakdown of the concept of absolute simultaneity. If two observers are in relative motion and attempt to synchronize their clocks by sending light signals out while accounting for the universal propagation speed c, then even if the clocks at the left end of the rods were set equal initially, the clocks at the right ends will not be synchronized when the signal arrives. Considering a series of events along the $x-$axis that are synchronized in the lab at $t = 0$, the "moving" observer would assert that, the farther the events are to the right, the earlier they are.

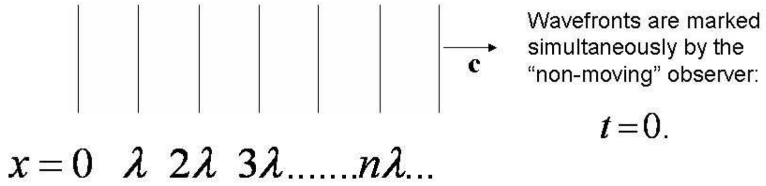

FIG. 5: The observer in the lab frame marks wave fronts simultaneously.

The relativity of simultaneity can be related directly to the Sagnac effect. Suppose that observers on earth's equator attempt to synchronize a system of clocks that are distributed around the equator, by sending a light signal all the way once around with a system of reflectors. The earth's equatorial radius is $a_1 = 6378.137$ m. The length of the equatorial circumference is $2\pi a_1$ and to observers on the rotating earth it appears that it takes the time $2\pi a_1/c$ for the signal to propagate once around. But to observers in a local, non-rotating frame, the starting point moves a distance $vt = \omega a_1 t = \omega a_1(2\pi a_1/c)$ during the process, where $\omega = 7.291151467 \times 10^{-5}$ rad/s is the angular velocity of earth's rotation. Thus the signal takes an extra amount of time

$$\Delta t = \frac{vt}{c} = \frac{2\omega}{c^2}\pi a_1^2 = \frac{2\omega A}{c^2} \tag{12}$$

to complete the circuit. The observers will disagree on the result of the synchronization process. The discrepancy, obtained from Eq. (11) by substituting $2\pi a_1$ for x and ωa_1 for v, is

$$t - t' = \frac{2\omega\pi a_1^2}{c^2}, \tag{13}$$

in agreement with Eq. (12).

Actually the entire discussion of the Sagnac effect can be skipped if the topic is not of interest. It is included here because of its intimate connection with the relativity of simultaneity, and its importance in the "common-view" method of comparison of remotely locately clocks.

V. FIRST-ORDER DOPPLER EFFECT

The relativity of simultaneity leads directly to another first-order effect, the Doppler effect. A thought experiment, illustrated in Figure 5, leads to the prediction of a frequency redshift of a receding source. Imagine the observer in the laboratory frame sends out a series of electromagnetic pulses or cycles of frequency f. In the lab these will separated by a wavelength $\lambda = c/f$ and the wavelength can be measured by marking the positions of the wavecrests simultaneously. However, to an observer moving with speed v in the direction of propagation, these position marks appear to be too early, the farther they are to the right, according to Eq. (11). For example, the wave crest numbered $n = 1$ in Figure 5 would appear to be marked too early by an amount $v\lambda/c^2$. The moving observer would have to allow this much extra time before marking the wavefront in order to obtain a correct measurement of wavelength in the moving frame. During this time, $v\lambda/c^2$, the wavefront moves an addition distance $c(v\lambda/c^2) = v\lambda/c$. Therefore to the moving observer the wavelength is

$$\lambda' = \lambda + \frac{v\lambda}{c} = \lambda(1 + \frac{v}{c}). \tag{14}$$

The wavelength is increased, so the frequency is decreased. The fractional frequency shift is

$$\frac{\Delta f}{f} = -\frac{v}{c}\,. \tag{15}$$

Thus the constancy of c, the relativity of simultaneity, and the first-order Doppler shift are intimately connected. One can likewise construct arguments such that from any two of these effects the third one follows. For example, given the first-order Doppler redshift of a receding source, the constancy of the speed of light, and the relation $c = f\lambda$, it is easy to invent a thought experiment to derive the relativity of simultaneity. The Doppler shift is important in the derivation of the gravitational frequency shift, given in the next section.

VI. GRAVITATIONAL FREQUENCY SHIFT

A. Accelerating Rocket Experiment

Figure 6 pictures a thought experiment that, together with the principle of equivalence, leads to the prediction of a gravitational frequency shift. A rocket with acceleration g upwards contains a transmitter and a receiver at height L above the transmitter. We view this accelerating rocket from a reference frame which is inertial. At the instant the rocket starts up, the signal is transmitted. The time required to reach the receiver is essentially $t = L/c$. But during the propagation of the signal to the receiver, the receiver picks up a velocity

$$v = gt = \frac{gL}{c}\,. \tag{16}$$

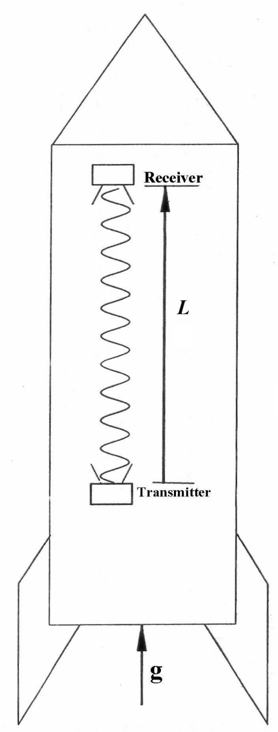

FIG. 6: A signal rising up in an accelerating rocket will be redshifted; the Principle of Equivalence then leads to a prediction of the gravitational frequency shift.

Then when the signal is detected the frequency must be the same as that detected by a receiver moving at constant velocity, at the location of the receiver, but not accelerating. To this receiver, the transmitter is a receding source and the frequency will be Doppler shifted downwards:

$$\frac{\Delta f}{f} = -\frac{v}{c} = -\frac{gL}{c^2}\,. \tag{17}$$

By the principle of equivalence, the physics in the accelerating rocket is the same as it would be in a gravitational field of strength g. Thus the accelerated observer would attribute the frequency change to the gravitational field, and then the product gL may be identified in terms of the change in gravitational potential, $\Delta\Phi = gL$.

$$\frac{\Delta f}{f} = -\frac{\Delta\Phi}{c^2}\,. \tag{18}$$

As the rocket continues to accelerate with constant velocity, (as long as the velocity does not get too large), the frequency shift will remain constant so that to the observer in the rocket, the frequency of the clock that is driving the transmitter, and that in this case is lower down in the gravitational field, is beating more slowly. So comparing two clocks at different potentials, they should beat at different rates. The rate difference between a clock at potential $\Phi + \Delta\Phi$ and one at Φ can be computed in terms of a fractional frequency shift by reversing the sign in Eq. (18):

$$\frac{\Delta f}{f} = \frac{\Delta \Phi}{c^2}\,. \tag{19}$$

B. Gravitational frequency shifts in the GPS

Consider then two clocks, a reference clock at rest on the earth's equator at radius a_1, and an atomic clock in orbit at radius r. The gravitational potential of the clock in orbit is, to sufficient accuracy

$$\Phi = -\frac{GM}{r} \tag{20}$$

where for earth, $GM = 3.986004415 \times 10^{14}$ m^3/s^2. The gravitational potential of the reference clock is affected slightly by the fact that the earth is not a perfect sphere, and a contribution from the quadupole potential is needed:

$$\Phi(\mathbf{r}) = -\frac{GM}{r}(1 - J_2\frac{a_1^2}{r^2}\frac{(3z^2 - r^2)}{2r^2})\,. \tag{21}$$

where $J_1 = 1.0863 \times 10^{-3}$ is earth's quadrupole moment coefficient. A clock on the equator at $z = 0$ then has the gravitational potential

$$\Phi_{\text{reference}} = -\frac{GM}{a_1}(1 + \frac{J_2}{2})\,. \tag{22}$$

The total gravitational frequency shift of GPS satellite clocks is therefore

$$\frac{\Delta f}{f} = \frac{\Phi - \Phi_{\text{reference}}}{c^2} = -\frac{GM}{c^2 r} - (-\frac{GM}{a_1 c^2}(1 + \frac{J_2}{2})) \approx 5.288 \times 10^{-10}\,. \tag{23}$$

This is actually a huge effect. If not accounted for, in one day it could build up to a timing error that would translate into a navigational error of 13.7 km. Good GPS satellite clocks have instrinsic stabilities that allow them to keep time to within a few parts in 10^{14} after a day. The gravitational frequency shift is thousands of times bigger than this.

VII. TIME DILATION

Another very important effect in the GPS is the relative slowing of moving clocks. In Figure 7 is a diagram of a simple thought experiment, based again on the constancy of the

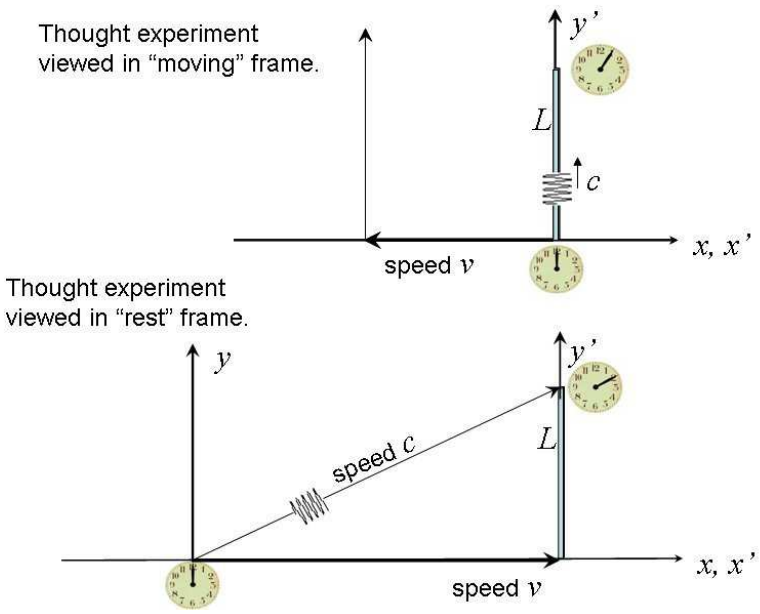

FIG. 7: Thought experiment leading to time dilation.

speed of light, that leads to a prediction of this effect. Imagine an experiment in which the "moving" observer lays a measuring rod of length L out along the $y'-$axis and transfers the time from a clock at $y' = 0$ to one at $y' = L$ by sending a light signal from the bottom of the rod to the top. The time required is

$$t' = L/c\,. \tag{24}$$

On the other hand, in the view of the "lab" observer, the light signal pulse goes out along the moving rod, so has an $x-$component of velocity equal to v. The light path follows the hypotenuse of a triangle with speed c, so the vertical component of velocity is

$$c_y = \sqrt{c^2 - v^2}\,. \tag{25}$$

Therefore it takes longer for the light to get to the end of the rod. The arrival time of the signal at the end of the rod is

$$t = \frac{L}{\sqrt{c^2 - v^2}}\,. \tag{26}$$

so the time of the "moving" clock at the top end of the rod, and the "rest" clock at the same location are related by

$$t' = \sqrt{1 - \frac{v^2}{c^2}}t\,. \tag{27}$$

Thus, a clock moving relative to a system of synchronized clocks in an inertial frame beats more slowly. The square root in Eq. (27) can be approximately expanded using the binomial theorem:

$$\sqrt{1-\frac{v^2}{c^2}} \approx 1-\frac{1}{2}\frac{v^2}{c^2}\,. \tag{28}$$

In the GPS, satellite velocities are close to 4000 m/s, so the order of magnitude of the time dilation effect is

$$-\frac{1}{2}\frac{v^2}{c^2} \approx -8.35\times 10^{-11} \tag{29}$$

This is also a huge effect. A reference clock on earth's equator is also in motion, but with a smaller speed, of order 465 m/s. To obtain the fractional frequency difference between a GPS satellite clock and a reference clock on the equator, we have to compute the difference:

$$\frac{\Delta f}{f} = -\frac{1}{2}\frac{v^2}{c^2} - (-\frac{1}{2}\frac{(\omega a_1)^2}{c^2}) = -8.228\times 10^{-11}\,. \tag{30}$$

If not accounted for, this would build up to contribute a navigational error of order 2.13 km/day.

Because of these frequency offsets, it is best to view the GPS satellite constellation and the reference clocks on the rotating earth from the point of view of the ECI frame. Reference clocks at any location, and moving arbitrarily, are assumed to be synchronized so that they read the same time as imaginary clocks at rest in the underlying inertial frame with which they momentarily coincide. Any reference clock can be chosen to establish the rate, or the length of the time unit. In practice, the ensemble of clocks at the U. S. Naval Observatory is adopted as the standard. These clocks are not on the equator, but essentially beat at the same rate as clocks on the equator. This is illustrated in Figure 8. The earth is slightly flattened, due to its rotation. The polar radius is less than the equatorial radius. Clocks on the surface of mean sea level that are closer to the equator beat more slowly due to time dilation. But they are higher up in the gravitational field and beat faster due to their greater gravitational potential. Together with the quadrupole potential, these effects compensate to very high precision. The earth's "geoid"–the surface at mean sea level–has the remarkable property that clocks at rest anywhere on this surface all beat at the same rate. Such reference clocks then all beat at the same rate. Corrections can be applied to atomic clocks that are not on the geoid so that they effectively beat at the rate of clocks on the geoid. (At NIST, the correction is -15.5 ns/day.)

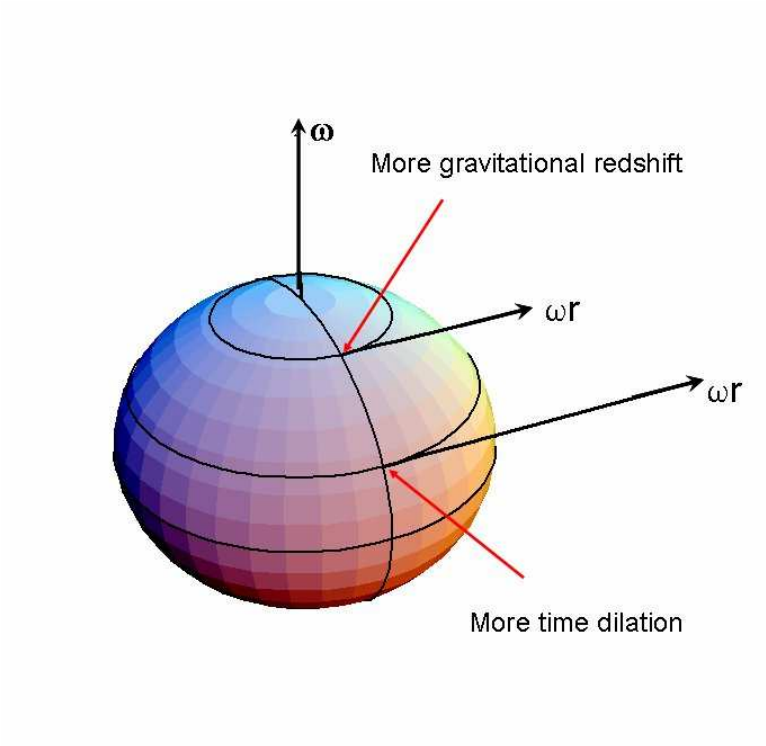

FIG. 8: Atomic clocks on earth's geoid beat at equal rates due to compensating relativistic effects.

VIII. NET RELATIVISTIC FREQUENCY SHIFT

A. Factory Frequency Offset

Since time dilation and gravitational frequency shifts are small compared to unity, their net contribution to the fractional frequency shift of a satellite clock, relative to one of the reference clocks, is obtained by simple addition. Combining Eqs. (23) and (30), the sum is

$$\frac{\Delta f}{f} = \frac{(-\frac{GM}{r} + \frac{GM}{a_1}(1 + \frac{J_2}{2}))}{c^2} + \frac{-\frac{1}{2}v^2 - (-\frac{1}{2}(\omega a_1)^2)}{c^2} \, . \tag{31}$$

There are thus a handful of relativistic effects, amounting to a total that is about 10,000 times too large to ignore.

The above expression is further modified by inserting the fact that the satellites are, to a good approximation, in Keplerian orbits in which their energy is conserved. It won't be shown here, but the study of a Keplerian orbit results in the following expression for the total energy per unit mass of the satellite

$$\frac{1}{2}v^2 - \frac{GM}{r} = -\frac{GM}{2a} \, , \tag{32}$$

where $a = 26,562$ km is the designed semi-major axis of a GPS satellite orbit. Then the velocity term in Eq. (31) can be eliminated in favor of r and a using Eq. (32), yielding the final expression

$$\frac{\Delta f}{f} = -\frac{2GM}{c^2}\left(\frac{1}{r} - \frac{1}{a}\right) - \frac{3GM}{2c^2 a} + \frac{GM}{c^2 a_1}\left(1 + \frac{J_2}{2}\right) + \frac{1}{2}\frac{(\omega a_1)^2}{c^2}\,. \tag{33}$$

The contributions are thus separated into a constant part, plus a small term that vanishes if the orbit is perfectly circular. The constant part is

$$-\frac{3GM}{2c^2 a} + \frac{GM}{c^2 a_1}\left(1 + \frac{J_2}{2}\right) + \frac{1}{2}\frac{(\omega a_1)^2}{c^2} = 4.4647 \times 10^{-10}\,. \tag{34}$$

In the older satellites these terms were compensated by setting the atomic clock frequencies down by this amount before launch–the so-called "factory frequency offset." Atomic clocks that have been recently launched are based on Rubidium atoms. These clock frequencies may be bumped during launch so they are measured after orbit insertion and the necessary frequency corrections are transmitted to the receivers in the navigation message.

B. Eccentricity Effect

The first term in Eq. (33) varies during the orbit. It is proportional to the orbit eccentricity and can give rise to an error of as much as 75 ns if not accounted for. The early satellites had limited computing power and the system was designed so that the receiver would apply this correction to the transmitted time. Thus every receiver is supposed to contain relativity software to apply this "eccentricity correction." Nowadays the satellites have plenty of computing capability and could easily apply this correction before transmitting their time signals. It is thought that, in the Soviet Union's GLONASS satellite system such corrections are applied in the satellite processor.

C. Orbit Adjustment Effects

When a satellite reaches the end of its life and is decomissioned, some other spare satellite in a parking orbit is placed into an appropriate slot so that the GPS system can continue operations with its full complement of satellites. Currently there are 29 satellites in orbit; 24 is the number nominally used in operations. Suppose the orbit of a satellite were perfectly circular. Then in Eq. (34) there is only one term that depends on the orbit, the first term that has the semi-major axis a in the denominator. If thrusters are fired so that a is changed to $a + \delta a$, then the fractional frequency will change by the amount

$$\delta\left(\frac{\Delta f}{f}\right) = +\frac{3GM\delta a}{2c^2a^2}\,. \tag{35}$$

For a typical altitude increase of 20 km, this change is

$$\delta\left(\frac{\Delta f}{f}\right) = 1.88 \times 10^{-13}\,. \tag{36}$$

This prediction works so well that when orbits are adjusted, the expected frequency shift is anticipated; it is not necessary to wait several days while the new frequency is measured from the ground, before placing the new satellite clock into operation.

IX. CONCLUSIONS

Relativistic effect in the GPS are far too large to ignore. Understanding these effects is readily accessible to undergraduates; a small number of thought experiments based on a few fundamental principles leads directly to correct predictions of these effects. It is not necessary to introduce the fundamental scalar, or a metric tensor, in this elementary treatment. Some things have been purposely glossed over, such as the possibility of Lorentz contraction (it actually is not important; the GPS is a timing system.) More detail can be found in the references.[1],[2] See Chapter 10 in Ref. [3].

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[1] N. Ashby, Living Reviews in Relativity, http://www.Relativity.LivingReviews.org/Articles/lrr2003-1 (2003).

[2] N. Ashby, Physics Today pp. 41–47 (2002).

[3] A. Ashtekar, *100 Years of Relativity* (World Scientific, 2006).