Using Dimensions

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Some of the most interesting results of hydrodynamics, such as the sixteen-fold increase in flow down a pipe on doubling the radius, can actually be found without doing any calculations, just from dimensional considerations.

We symbolize the "dimensions" *mass*, *length* and *time* by *M*, *L*, *T*. We then write the dimensions of other physical quantities in terms of these. For example, velocity has dimensions LT^{-1}, and acceleration LT^{-2}.

We shall use *square brackets* [] to denote the dimensions of a quantity, for example, for velocity, we write $[v] = LT^{-1}$. Force must have the same dimensions as mass times acceleration, so $[F] = MLT^{-2}$. This "dimensional" notation does *not* depend on the units we use to measure mass, length and time.

<p style="text-align:center; color:red;">***All equations in physics must have the same dimensions on both sides.***</p>

We can see from the equation defining the coefficient of viscosity η, $F/A = \eta v_0 / d$, (the left hand side is force per unit area, the right hand v_0/d is the velocity gradient) that

$$[\eta] = ([F]/[A]) \cdot [d]/[v] = \left(MLT^{-2}/L^2 \right) \cdot L / LT^{-1} = ML^{-1}T^{-1}.$$

How can thinking dimensionally help us find the flow rate *I* through a pipe? Well, the flow itself, say in cubic meters per second, has dimensions $[I] = M^3 T^{-1}$. What can this flow depend on?

The physics of the problem is that the pressure difference ΔP between the ends of the pipe of length L is doing work overcoming the viscous force. The *only* parameters determining the flow are therefore: the pressure gradient, $\Delta P / L$, the viscosity η, and the radius of pipe cross section *a*. Note here that we are assuming the flow is *steady*—no acceleration—so the mass, or more precisely density, of the fluid plays no part. Of course, if the flow is downward, the density has an *indirect* role in that the weight of the fluid generates the pressure gradient, but we've already included the pressure as a parameter.

Therefore,

$$\text{Flow } I = f\left(\Delta P / L, \eta, a \right)$$

where *f* is some function we don't know, but we *do* know that the two sides of this equation must match dimensionally, so *f* must have the same dimensions as *I*, that is, $L^3 T^{-1}$.

Now ΔP, a pressure, has dimensions $[F]/[A] = MLT^{-2}L^{-2}$ so $\Delta P/L$ has dimensions $ML^{-2}T^{-2}$.

The other variables in *f* have dimensions $[\eta] = ML^{-1}T^{-1}$ (from above) and $[a] = L$.

The game is to put these three variables (or powers of them) together to give a function *f* having the dimensions of flow, that is, L^3T^{-1}, otherwise the above equation must be invalid.

The first thing to notice is that there is no *M* term in flow, and none in *a* either, so $\Delta P/L$ and η must appear in the equation in such a way that their *M* terms cancel, that is, one divides the other.

We know of course that increased pressure increases the flow, so they must appear in the combination $\Delta P/L\eta$. This gets rid of *M*. The next task is to put this combination, which has itself dimensions $MLT^{-2}L^{-2}/ML^{-1}T^{-1} = L^{-1}T^{-1}$, together with $[a] = L$, to get a quantity with the dimensions of flow, L^3T^{-1}. The unique choice is to multiply $\Delta P/L\eta$ by a^4.

We therefore conclude that the flow rate through a circular pipe must be given by:

$$I = C(\Delta P/L\eta)a^4.$$

This is certainly much easier than solving the differential equation and integrating to find the flow rate! The catch is the unknown constant *C* in the equation—we can't find that without doing the hard work. However, we *have* established from this dimensional argument that the flow rate increases by a factor of 16 when the radius is doubled.

It should be noted that this conclusion *does* depend on the validity of the assumptions made—in particular, that the flow is uniform and in straight streamlines. At sufficiently high pressure, the flow becomes turbulent. When this happens, the pressure causes the fluid to bounce around inside the pipe, and the flow pattern will then depend also on the *density* of the fluid, which was irrelevant for the slow laminar flow, and the reasoning above will be invalid.

Exercise: derive the depth dependence of the steady flow of a wide river under gravity. (Note: The appropriate flow rate is cubic meters per second *per meter of width of the river*.)

So dimensional analysis cannot give overall dimensionless constants, but can predict how flow will change when a physical parameter, such as the pressure or the size of the pipe, is altered. We've shown above how it rather easily gives a nonobvious result, the a^4 dependence of flow on radius, which we found earlier with a good deal of work. But as we shall see, dimensional analysis can also illuminate the essential physics of flow problems where exact mathematical analysis is far more difficult, such as Stokes' Law in the next lecture, and help us understand how the nature of fluid flow changes at high speeds.