The Number *e* and the Exponential Function

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Disclaimer: these notes are not mathematically rigorous. Instead, they present quick, and, I hope, plausible, derivations of the properties of e, e^x and the natural logarithm.

The Limit $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$

Consider the following series: $(1+1)$, $(1+\frac{1}{2})^2$, $(1+\frac{1}{3})^3$, ..., $(1+\frac{1}{n})^n$,... where *n* runs through the positive integers. What happens as *n* gets very large?

It's easy to find out if you use a scientific calculator having the function x^y. The first three terms are 2, 2.25, 2.37. You can use your calculator to confirm that for n = 10, 100, 1000, 10,000, 100,000, 1,000,000 the values of $(1+\frac{1}{n})^n$ are (rounding off) 2.59, 2.70, 2.717, 2.718, 2.71827, 2.718280. These calculations strongly suggest that as *n* goes up to infinity, $(1+\frac{1}{n})^n$ goes to a definite limit. It can be proved mathematically that $(1+\frac{1}{n})^n$ *does* go to a limit, and this limiting value is called *e*. The value of *e* is 2.7182818283… .

To try to get a bit more insight into $(1+\frac{1}{n})^n$ for large *n*, let us expand it using the binomial theorem. Recall that the binomial theorem gives all the terms in $(1 + x)^n$, as follows:

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+...+x^n$$

To use this result to find $(1+\frac{1}{n})^n$, we obviously need to put $x = 1/n$, giving:

$$(1+\tfrac{1}{n})^n = 1+n.\tfrac{1}{n}+\frac{n(n-1)}{2!}(\tfrac{1}{n})^2+\frac{n(n-1)(n-2)}{3!}(\tfrac{1}{n})^3+....$$

We are particularly interested in what happens to this series when *n* gets very large, because that's when we are approaching *e*. In that limit, $n(n-1)/n^2$ tends to 1, and so does $n(n-1)(n-2)/n^3$. So, for large enough *n*, we can ignore the *n*-dependence of these early terms in the series altogether!

When we do that, the series becomes just:

$$1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+...$$

And, the larger we take n, the more accurately the terms in the binomial series can be simplified in this way, so as n goes to infinity this simple series represents the limiting value of $(1+\frac{1}{n})^n$. *Therefore, e must be just the sum of this infinite series*.

(Notice that we can see immediately from this series that e is less than 3, because 1/3! is less than $1/2^2$, and 1/4! is less than $1/2^3$, and so on, so the whole series adds up to less than $1 + 1 + ½ + 1/2^2 + 1/2^3 + 1/2^4 + \ldots = 3$.)

The Exponential Function e^x

Taking our definition of e as the infinite n limit of $(1+\frac{1}{n})^n$, it is clear that e^x is the infinite n limit of $(1+\frac{1}{n})^{nx}$. Let us write this another way: put $y = nx$, so $1/n = x/y$. Therefore, e^x is the infinite y limit of $(1+\frac{x}{y})^y$. The strategy at this point is to expand this using the binomial theorem, as above, and get a power series for e^x.

(Footnote: there is one tricky technical point. The binomial expansion is only simple if the exponent is a whole number, and for general values of x, $y = nx$ won't be. But remember we are only interested in the limit of very large n, so if x is a rational number a/b, where a and b are integers, for n any multiple of b, y *will* be an integer, and pretty clearly the function $(1+\frac{x}{y})^y$ is continuous in y, so we don't need to worry. If x is an irrational number, we can approximate it arbitrarily well by a sequence of rational numbers to get the same result.)

So, we need to do the binomial expansion of $(1+\frac{x}{y})^y$ where y is an integer—to make this clear, let us write $y = m$.

$$(1+\tfrac{x}{m})^m = 1 + m.\tfrac{x}{m} + \frac{m(m-1)}{2!}(\tfrac{x}{m})^2 + \frac{m(m-1)(m-2)}{3!}(\tfrac{x}{m})^3 + \ldots$$

Notice that this has exactly the same form as the binomial expansion of $(1+\frac{1}{n})^n$ in the paragraph above, except that now a power of x appears in each term. Again, we are only interested in the limiting value as m goes to infinity, and in this limit $m(m-1)/m^2$ goes to 1, as does $m(m-1)(m-2)/m^3$. Thus, as we take m to infinity, the m dependence of each term disappears, leaving

$$e^x = \lim_{m\to\infty}(1+\tfrac{x}{m})^m = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Differentiating e^x

$$\frac{d}{dx}e^x = \frac{d}{dx}(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots) = 1 + x + \frac{x^2}{2!} + \ldots$$

so when we differentiate e^x, we just get e^x back. This means e^x is the solution to the equation $\frac{dy}{dx} = y$, and also the equation $\frac{d^2y}{dx^2} = y$, etc. More generally, replacing x by ax in the series above gives

$$e^{ax} = 1 + ax + \frac{a^2x^2}{2!} + \frac{a^3x^3}{3!} + \ldots$$

and now differentiating the series term by term we see $\frac{d}{dx}e^{ax} = ae^{ax}$, $\frac{d^2}{dx^2}e^{ax} = a^2e^{ax}$, etc., so the function e^{ax} is the solution to differential equations of the form $\frac{dy}{dx} = ay$, or of the form $\frac{d^2y}{dx^2} = a^2y$ and so on.

Instead of differentiating term by term, we could have written

$$\frac{d}{dx}e^{ax} = \lim_{\Delta x \to 0} \frac{e^{a(x+\Delta x)} - e^{ax}}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^{ax}\left(e^{a\Delta x} - 1\right)}{\Delta x} = ae^{ax}$$

where we have used $\left(e^{a\Delta x} - 1\right) \to a\Delta x$ in the limit $\Delta x \to 0$.

The Natural Logarithm

We define the natural logarithm function $\ln x$ as the inverse of the exponential function, by which we mean

$$y = \ln x, \text{ if } x = e^y$$

Notice that we've switched x and y from the paragraph above! Differentiating the exponential function $x = e^y$ in this switched notation,

$$\frac{dx}{dy} = e^y = x, \text{ so } \frac{dy}{dx} = \frac{1}{x}.$$

That is to say,

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

Therefore, $\ln x$ can be written as an integral,

$$\ln x = \int_1^x \frac{dz}{z}.$$

You can check that this satisfies the differential equation by taking the upper limit of the integral to be $x+\Delta x$, then x, subtracting the second from the first, dividing by Δx, and taking Δx very small. But why have I taken the lower limit of the integral to be 1? In solving the differential equation in this way, I could have set the lower limit to be any constant and it would still be a solution—but it would not be the inverse function to e^y unless I take the lower limit 1, since that gives for the value $x = 1$ that $y = \ln x = 0$. We need this to be true to be consistent with $x = e^y$, since $e^0 = 1$.

Exercise: show from the integral form of $\ln x$, that for small x, $\ln(1 + x)$ is approximately equal to x. Check with your calculator to see how accurate this is for $x = 0.1$, 0.01.