Hydrostatics: from Archimedes to Jefferson

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High Tech Crime Detection, Version 1.0

We begin this lecture in Syracuse, Sicily, 2200 years ago, with Archimedes and his friend king Heiro. The following is quoted from Vitruvius, a Roman engineer and architect writing just before the time of Christ:

> *Heiro, after gaining the royal power in Syracuse, resolved, as a consequence of his successful exploits, to place in a certain temple a golden crown which he had vowed to the immortal gods. He contracted for its making at a fixed price and weighed out a precise amount of gold to the contractor. At the appointed time the latter delivered to the king's satisfaction an exquisitely finished piece of handiwork, and it appeared that in weight the crown corresponded precisely to what the gold had weighed.*
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> *But afterwards a charge was made that gold had been abstracted and an equivalent weight of silver had been added in the manufacture of the crown. Heiro, thinking it an outrage that he had been tricked, and yet not knowing how to detect the theft, requested Archimedes to consider the matter. The latter, while the case was still on his mind, happened to go to the bath, and on getting into a tub observed that the more his body sank into it the more water ran out over the tub. As this pointed out the way to explain the case in question, without a moments delay and transported with joy, he jumped out of the tub and rushed home naked, crying in a loud voice that he had found what he was seeking; for as he ran he shouted repeatedly in Greek, "Eureka, Eureka."*
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> *Taking this as the beginning of his discovery, it is said that he made two masses of the same weight as the crown, one of gold and the other of silver. After making them, he filled a large vessel with water to the very brim and dropped the mass of silver into it. As much water ran out as was equal in bulk to that of the silver sunk in the vessel. Then, taking out the mass, he poured back the lost quantity of water, using a pint measure, until it was level with the brim as it had been before. Thus he found the weight of silver corresponding to a definite quantity of water.*
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> *After this experiment, he likewise dropped the mass of gold into the full vessel and, on taking it out and measuring as before, found that not so much water was lost, but a smaller quantity: namely, as much less as a mass of gold lacks in bulk compared to a mass of silver of the same*

> *weight. Finally, filling the vessel again and dropping the crown itself into the same quantity of water, he found that more water ran over for the crown than for the mass of gold of the same weight. Hence, reasoning from the fact that more water was lost in the case of the crown than in that of the mass, he detected the mixing of silver with the gold and made the theft of the contractor perfectly clear.*

What is going on here is simply a measurement of the *density*—the mass per unit volume—of silver, gold and the crown. To measure the masses some kind of scale is used, note that at the beginning a precise amount of gold is weighed out to the contractor. Of course, if you had a nice rectangular brick of gold, and knew its weight, you wouldn't need to mess with water to determine its density, you could just figure out its volume by multiplying together length, breadth and height in meters, and divide the mass, or weight, in kilograms, by the volume to find the density in kilograms per cubic meter, or whatever units are convenient. (Actually, the original metric density measure was in grams per cubic centimeter, with the nice feature that the density of water was exactly 1, because that's how the gram was defined (at 4 degrees Celsius and atmospheric pressure, to be absolutely precise). In these units, silver has a density of 10.5, and gold of 19.3. We shall be using the standard MKS units, so water has a density $1000 kg/m^3$, silver $10,500 kg/m^3$, etc.

The problem with just trying to find the density by figuring out the volume of the crown is that it is a very complicated shape, and although one could no doubt find its volume by measuring each tiny piece and calculating a lot of small volumes which are then added together, it would take a long time and be hard to be sure of the accuracy, whereas lowering the crown into a filled bucket of water and measuring how much water overflows is obviously a pretty simple procedure. (You do have to allow for the volume of the string!). Anyway, the bottom line is that if the crown displaces more water than a block of gold of the same weight, the crown isn't pure gold.

Actually, there is one slightly surprising aspect of the story as recounted above by Vitruvius. Note that they had a weighing scale available, and a bucket suitable for immersing the crown. Given these, there was really no need to measure the amount of water slopping over. All that was necessary was to weigh the crown under water, then dry it off and weigh it out of the water. By Archimedes' Principle, the difference in these weights is equal to the weight of water displaced. This is definitely a less messy procedure--there is no need to fill the bucket to the brim in the first place, all that is necessary is to be sure that the crown is fully immersed, and not resting on the bottom or caught on the side of the bucket, during the weighing.

Of course, maybe Archimedes had not figured out his Principle when the king began to worry about the crown, perhaps the above experiment led him to it. There seems to be some confusion on this point of history.

We now turn to a discussion and derivation of Archimedes' Principle. To begin with, it is essential to understand clearly the concept of *pressure* in a fluid.

Pressure

Perhaps the simplest way to start thinking about pressure is to consider pumping up a bicycle tire with a hand pump. Pushing in the handle compresses the air inside the cylinder of the pump, raising its pressure—it gets more difficult to compress further. At a certain point, the air is compressed enough that it can get through a valve into the tire. Further pushing on the handle transfers the air to the tire, after which the handle is pulled back and the pump refills with outside air, since the valve on the tire is designed not to allow air to flow out of the tire. The outside air gets into the pump because there is, effectively, another valve—pushing the handle down pushes a washer down the inside of the cylinder to push the air out. The flexible rubber washer has a metal disk behind it so that it cannot bend far enough backwards to let air past it. On the return stroke, the washer has no rigidity, and the disk is on the wrong side for purposes of keeping it stiff, so it bends to allow air to get around it into the cylinder. As you continue to inflate the tire, it gets harder, and it’s not just that you’re wearing out. As the pressure in the tire increases, you have to compress the air in the pump more and more before it reaches the pressure where the valve on the tire opens and lets it in. This is hardly surprising, because the pressure inside the tire is building up, and the valve is not going to open until the pressure from the pump has built up beyond that in the tire, so that the new air can push its way into the tire.

If you look at the bicycle tire, you will probably see written on it somewhere the appropriate pressure for riding, in pounds per square inch. A gas, for instance air, under pressure pushes outwards on all the walls containing it with a steady force, equal areas of the walls feeling an equal force which is proportional to the area, hence the units, pounds per square inch. A typical ten-speed tire might be eighty pounds per square inch. (In metric units, one pound per square inch is about 6900 Newtons per square meter, so a typical ten-speed tire has pressure around 5.5×10^5 Newtons per square meter. A useful way to remember this equivalence is that the pressure of the atmosphere, which is about 15 pounds per square inch, is about 10^5 Newtons per square meter.) This means, if your pump has a cylinder with an internal cross-sectional area of one square inch, you must push it with a force of better than eighty pounds to get air into it when it’s fully inflated. Of course, a pump with a narrower cylinder (which most have!) will need a proportionately smaller force, but then you will get less air delivered per stroke.

You can measure the air pressure in a tire using a pressure gauge. The basic idea of such a gauge is that on holding it against the tire valve, the valve opens and lets out air into a small cylinder with a moveable end, this end feels a force equal to the pressure in the tire (which is also the pressure in this cylinder) multiplied by its cross sectional area. The moveable end can push against a spring, with some device to lock it when it reaches maximum compression of the spring, to make it easy to read.

We can construct a more primitive, but easy to understand, pressure measuring device by putting water in a U-shaped tube. This is called a manometer. Suppose we leave one end of the tube open to the air but connect a partially inflated balloon to the other end. We observe that the pressure from the balloon pushes the water down on its side, and hence the water rises on the open side. Suppose the water goes down 0.1m on the balloon side, so it goes up 0.1m on the other side. What is the pressure in the balloon? It is clearly enough to support the weight of a column of water 0.2m high. To find out what this pressure is, we need to know how heavy water is. A cubic meter of water weighs 1000kg. Imagine this cubic meter in a cubic bucket, with a square bottom, one meter by one meter. That means the base has an area of one square meter, and this area is bearing the weight of the water, 1000kg, so the pressure at the base is approximately 10,000 Newtons per square meter. Now, our manometer, with a difference in water column heights of 0.2m, must be indicating a pressure of 2,000 Newtons per square meter. Clearly, this manometer is not going to be a handy device for measuring pressure in a ten-speed tire!

As you inflate a balloon, the elastic stretches more and more tightly, and it is this tension that balances the excess pressure of the air inside the balloon. You can feel the corresponding tightness in your own body by breathing deeply, and thus increasing the pressure in your lungs. Another way to feel increased pressure is to dive into deep water. The pressure under water increases with depth. We can easily demonstrate this with a rubber sheet stretched over a funnel on the end of a flexible rubber hose connected to the manometer. As we immerse the funnel in water, and push it in deeper, we see the pressure in it is rising, as measured by the manometer. In fact, we would expect to find that if we put it under six inches of water, the extra pressure measured will be given by an extra six inches height difference in the manometer arms. That would be true if we had a more accurate measuring instrument, but with our rather primitive arrangement the rubber sheet tension itself affects the reading slightly: as the rubber is pushed into the funnel, its own elastic contribution to the pressure inside varies. Fortunately, this turns out to be a rather small effect for the pressures we are looking at with the balloons we are using, so we ignore it.

The main point to notice is that the pressure increases *linearly* with depth, just as you might expect, because it's just the weight of water above you, weighing down on you, that's causing the pressure. But it is crucial to realize that *pressure in a fluid*, unlike the weight of a solid, *presses in all directions*. Consider again the "cubic meter" bucket filled with water we discussed previously. The pressure on the bottom was about one and a half pounds per square inch. If the water freezes to a block of ice, the weight of the ice on the bottom will exert the same pressure. However, consider now a small area on one of the side walls close to the bottom. The water presses on that too. If you're not sure about that, think what would happen if a small hole were to be bored into the side wall near the bottom. But if the water is a frozen block of ice, you could remove all the side walls, and the ice would stay in place. So fluids really are different in the way they push against containers, or against objects immersed in them.

A more vivid impression of the increase of pressure with depth, and the way it presses something in from all sides, is given by watching submarine movies, like the German das Boot, "The Boat". The submarine is forced to dive deep to avoid depth charges. As it goes down, rivets begin to pop, and water spurts in through the tiny holes. It comes in just as vigorously through a hole in the floor than through a hole in the ceiling--in fact more so, because of the greater depth. The pressure really is all around. How can fish survive under these circumstances? The main difference from this point of view between fish and submarines is that the submarine is hollow, and the pressure inside is kept down to a level tolerable to humans. The rivets pop because of the *difference* in pressure on the two sides of the sheets of steel forming the hull, which begins to buckle. In the fish, on the other hand, the fluids inside are at the same pressure as those outside. Of course, a piece of solid flesh of the fish feels this pressure all around, but it takes far greater pressure to compress a piece of solid flesh in a damaging way. (Flesh is mostly water anyway, and water is almost incompressible). For a human diver to go to great depths, the gas in the lungs must be adjusted to a pressure approximately matching the surroundings. This can be done. The problem that arises is that at high pressures, nitrogen more readily dissolves in human blood. As the diver comes back towards the surface, the pressure drops, and the nitrogen reappears in the bloodstream as small bubbles—exactly the same phenomenon as the bubbles that appear in a soda bottle when the top is loosened. The bubbles can interrupt the blood circulation, causing great pain (the "bends") and can be fatal. So the return to surface must be gradual, to avoid a sudden appearance of large numbers of bubbles. Fish are more careful about changing depth, but apparently fish caught at considerable depth and pulled up to the surface get the bends.

Buoyancy

Consider now an object totally immersed in water, for example, a submarine. The water is pressing on its surface on all sides, top and bottom. Notice that the pressure on the bottom of the submarine (which is pressing it upwards) will be *greater* than the pressure on the top (pressing it down) just because the bottom is deeper into the water than the top is, and, as discussed above, the pressure increases linearly with the depth. Thus the total effect of the pressure forces is to tend to *lift* the submarine. This is called *buoyancy*.

To figure out how strong this buoyancy force is, imagine replacing the submarine by a *ghost* submarine—a large plastic bag filled with water, the plastic itself being extremely thin, and of negligible weight. The bag has the same size and shape as the submarine and is placed at the same depth in the water. The plastic is not stretched like a balloon, its natural size is just that of the submarine. Then this bag must feel the same pressure on each part of its surface as that on the corresponding bit of the submarine, since pressure only depends on depth, so the total buoyancy force on the bag must be the same as that on the submarine. But this bag is not going to rise or fall in the water, because it's really just part of the water--the bag can be as thin as you like, and we could even choose a plastic having the same density as water. Since it doesn't move, *the buoyancy force pushing it upwards must be just balanced by the weight of the water in the bag*. But we said that the submarine felt the same buoyancy force, so the upward force felt by the submari ne is *also* equal to the weight of a volume of water equal to the volume of the submarine.

This is the famous **Principle of Archimedes**:

A body immersed, or partially immersed, in a fluid is thereby acted on by an upward force of buoyancy equal to the weight of the fluid displaced.

This means that the weight of a boat, for example, must be equal to the weight of a volume of water equal to the volume of the part of the boat below the water line.

Galileo and Archimedes' Principle

Galileo fully appreciated how important Archimedes' Principle was in really understanding falling bodies of different weights, falling through media of different densities. In fact, he used it to great effect (page 66 on, Two New Sciences) to demolish Aristotle's assertion that a body ten times heavier will fall ten times faster, irrespective of the medium. Of course, this is all a little unfair to Aristotle, since Archimedes enunciated the Principle about a century after Aristotle died. The main point, which Galileo fully appreciated, is that the *weight* of a body, which is the force causing the constant downward acceleration, *must be reduced by the buoyancy force*, so the actual *total* downward force is the weight of the body minus the weight of an equal volume of the fluid. To quote Galileo (page 67, TNS):

Thus, for example, an egg made of marble will descend in water one hundred times more rapidly than a hen's egg, while in air falling from a height of twenty cubits the one will fall short of the other by less than four finger-breadths.

For this to be true, and no doubt Galileo did the experiment, the hen's egg must be about one per cent heavier than water.

Furthermore, Galileo fully realized that the same effect must be taking place in air, but the effect there is much less dramatic in general, because air (at sea level) has a density only 1/800 that of water. (Galileo thought it was 1/400). So for most objects the buoyancy force is tiny compared to the weight. The only exceptions are the bladders discussed by Galileo, in other words, balloons.

A further complication that must be borne in mind when thinking about the differing buoyancy forces in different media, and their effect on rates of fall, is that the different media also resist the motion by differing amounts. These are two quite different effects. The *buoyancy* force acts on the body even if it is at rest--it's the force that keeps ships afloat. In contrast, the *resistance*--air resistance or water resistance--is essentially *dynamic* in character, and does not depend particularly on the density of the fluid, although, of course liquids resist motion more than gases. But within those groups there are wide variations. For example, a small steel ball, say, falling through olive oil at room temperature will encounter a resistive force almost one hundred times stronger than that felt falling the same speed through water, yet olive oil is lighter than water.

To return to the buoyancy force in air, since it is about 1/800 that in water, and you are almost the same density as water, your weight as measured on a bathroom scales is lighter by about one part in 800 than your true weight. This is of course insignificant, but gives some idea how big a balloon filled with a lighter-than-air gas is needed to lift a few people. The lightest gas in nature is hydrogen, about 1/14 the density of air. Unfortunately, it is highly flammable, and early airships sometimes met a fiery end. The next lightest is helium, twice as heavy as hydrogen, but chemically completely unreactive. This is the gas used in the Goodyear blimp.

An extremely cheap alternative is just to use hot air---you can have a big bag, open at the bottom, with a heater underneath the opening. The tricky point here is the hotter the air is, the better it works, but you don't want to set the balloon on fire! Air raised to, say, 300F, has a density about two-thirds that of air at 70F, so the buoyancy you can get with hot air is substantially below that from helium. This means the balloon has to be a lot bigger to lift the same weight. Notice also that there is a limit to how high a balloon can get. The atmosphere gets thinner with height. This means that the weight of air displaced, and hence the buoyancy force, also decreases with height, so a balloon of given density can only reach a certain height. This can be partly compensated by making a balloon of easily stretched material, so as it goes up the gas pressure inside it expands it to greater size against the lessening outside air pressure, so increasing the buoyancy.

Living in an Ocean

We discussed earlier how fish living deep in the ocean adjusted to their high pressure environment by having equal pressure, essentially, throughout their bodies, so there were no stresses, in contrast to a submarine, which has lower pressure inside than out, and so a tendency to implode. The fish are doubtless quite unaware of the fact that they live in a high pressure environment, although intelligent ones might begin to figure it out if they saw enough submarines implode. Fish control their depth by slight changes in buoyancy, achieved by moving gas in and out of an air sac, using various mechanisms: for example, by changing the acidity of the bloodstream, so that the solubility of oxygen in the blood varies.

There is actually an analogy here to our own environment. We live at the bottom of an ocean of air that covers the entire planet. Although the consequent pressure is a lot less than that at the bottom of the Atlantic, it is by no means negligible. If we pump the air out of an ordinary aluminum can it will implode. The actual pressure is about fifteen pounds per square inch. If we think of water in a U-shaped manometer tube, open to the atmosphere on both sides, the air pressure is of course equal on the water in the two arms, and the levels are the same. If we now use a pump to remove the air above the water on one side, the pressure on the water lessens, and the continuing pressure in the other arm is no longer balanced out, so the water begins to rise in the arm with less air. This is just the phenomenon of suction, the same as drinking through a straw. The important thing to see is that it is the *outside* ambient air pressure that forces the liquid up the straw.

Once it is clear that suction simply amounts to removing air pressure, and thus allowing the external air pressure of fifteen pounds per square inch, to push liquid up a pipe, it is clear that there is a limit to what suction can achieve. As the water climbs in the pipe, the pressure at the bottom of the pipe from the column of water itself increases. Eventually it will balance off the air pressure, so even if there is a perfect vacuum above the water, it won't rise any higher. It turns out that the height of a column of water that produces a pressure at its base of fifteen pounds per square inch is about thirty feet.

Galileo was actually aware of this effect, but he did not realize it was a result of limited external pressure, he thought it arose from a limit on the strength of the suction attraction holding the water together. From page 16 of TNS,

> *I once saw a cistern (a well) which had been provided with a pump under the mistaken impression that the water might thus be drawn with less effort or in greater quantity than by means of the ordinary bucket. The stock of the pump carried its sucker and valve in the upper part so that the water was lifted by attraction and not by a push as is the case with pumps in which the sucker is placed lower down. This pump worked perfectly so long as the water in the cistern stood above a certain level; but below this level the pump failed to work. When I first noticed this phenomenon I thought the machine was out of order; but the workman whom I called in to repair it told me the defect was not in the pump but in the water which had fallen too low to be raised through such a height; and he added that it was not possible, either by a pump or by any other machine working on the principle of attraction, to lift water a hair's breadth above eighteen cubits; whether the pump be large or small this is the extreme limit of the lift. Up to this time I had been so thoughtless that, although I knew a rope, or rod of wood, or of iron, if sufficiently long, would break by its own weight when held by the upper end, it never occurred to me that the same thing would happen, only much more easily, to a column of water.*

The problem with this explanation of what held solids together, as Galileo went on to admit, was that a copper wire hundreds of feet long can be hung vertically without breaking, and it is difficult to see how suction can be that much more effective for copper. We now know he was on the wrong track for once, what holds solids together is electrical forces between atoms.

Barometers

The approximately fifteen pounds per square inch pressure of the atmosphere is the pressure measured by a barometer. A column of water in a pipe with the air removed from above it would be a perfectly good barometer, just not a very handy size, since it would be about thirty feet long, from the discussion above. The obvious way to improve on this is to use a liquid heavier than water, so a less high column of it exerts the same pressure. The liquid of choice is mercury, or quicksilver, which is 13.6 times heavier than water, so we need a pipe less than three feet long. The first barometer was made by a pupil of Galileo's, Evangelista Torricelli, and he explained how it worked in a letter written in 1644, two years after Galileo died. He wrote that the fluid rose in the pipe because of "the weight of the atmosphere", and "we live at the bottom of a sea of elemental air, which by experiment undoubtedly has weight".

The idea of using the barometer to measure heights first occurred to a Frenchman, Blaise Pascal, a few years later. He wrote an account in 1648 of taking the barometer up and down hills and tall buildings and measuring the difference in the height of the column of mercury. Of course, this is a bit tricky because the height of the column also varies with the weather, as the atmosphere sloshes about the actual amount of air above a particular point varies.

Just over a century later, on July 5, 1776, to be precise, Thomas Jefferson bought a barometer made in London at Sparhawk's, a shop in Philadelphia, which he happened to be visiting. On September 15, 1776, he found the height of the mercury to be 29.44 inches at Monticello, 30.06 inches at the tobacco landing on the Rivanna, and 29.14 inches at the top of Montalto. He used a table published in London in 1725 to translate these differences into heights, and concluded that Montalto was 280 feet above Monticello, and 792 feet above the Rivanna.

Checking Archimedes' Principle

Place a beaker of water about three-quarters full on a spring balance. Note the reading. Now dip your hand in the water, not touching the beaker, until the water reaches the top of he beaker. Note how much the measured weight has increased.

Question 1: If instead of putting your hand in the beaker, you had simply poured in extra water to fill it to the brim, would the measured weight have increased by the same amount, or more, or less?

Question 2: Suppose instead of dipping in your hand, you had immersed a piece of iron, (again not touching the beaker with the iron, of course) until the water reached the brim. Would this give a different reading? What if you had used a small balloon?

Take a piece of metal hanging from a spring. Note its weight, then weigh it under water in the beaker. Note also the change in weight on the spring scale under the beaker. Are these readings related? Now lower the piece of metal until it is resting on the bottom of the beaker. What do the springs read now? What happened to the buoyancy? Suppose there was a tiny spring scale on the bottom of the beaker, under the water, and the piece of metal was resting on it. What would it read?

Cartesian divers are essentially inverted test tubes, or other small containers, with trapped air inside, so that they have an overall density very close to that of water. You can put one in a container so that you can change the pressure, such as a plastic bottle with the top screwed on. If you increase the pressure, the diver dives. If you now release the pressure, it comes back up. Why?

Further comments: Carefully distinguishing between mass and density is nontrivial—after all, it took Archimedes to figure it out! Also, the units are a bit confusing. The simple everyday units would be pounds per cubic foot. The official International Units for scientific work these days are the MKS system, in which the standard length is the meter, just over a yard, and the standard weight is the kilogram, 2.2 pounds. The official unit of volume is then the cubic meter, about what a small pickup truck can carry, not a real handy size! The old official metric unit of volume was the cubic centimeter, and that was the amount of water that weighed one gram. In the new system, then, the new unit of volume is *one million* times bigger, and one cubic meter of water weighs 1,000 kilograms, which is one ton! A more handy unit of volume is the *liter*, 1,000 cubic centimeters (cc's), and a liter of water weighs one kilogram.

Question: As you know from floating in a pool, the human body has a density close to that of water. You know your own weight. What is your volume?

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