Lectures on Oscillations and Waves

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From a Circling Complex Number to the Simple Harmonic Oscillator

(*A review of complex numbers is provided in the appendix to these lectures.*)

Describing Real Circling Motion in a Complex Way

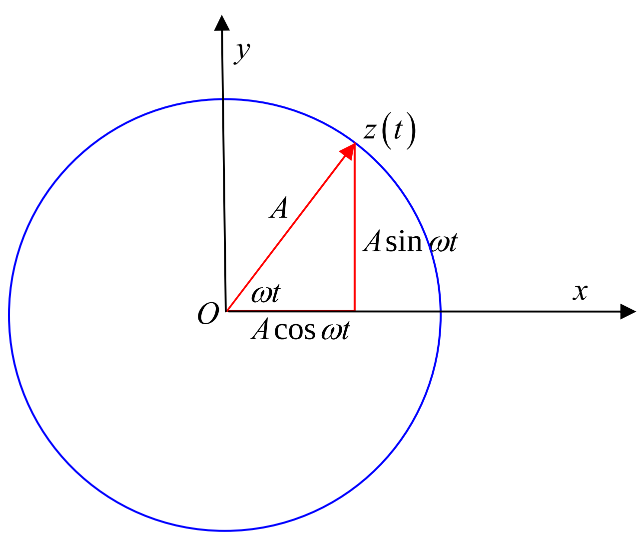
We've seen that any complex number can be written in the form $z = re^{i\theta}$, where r is the distance from the origin, and θ is the angle between a line from the origin to z and the x-axis. This means that if we have a set of numbers all with the same r, but different θ's, such as $re^{i\alpha}$, $re^{i\beta}$, etc., these are just different points on the circle with radius r centered at the origin in the complex plane.

Now think about a complex number that moves as time goes on: $z(t) = Ae^{i\omega t}$.

At time t, $z(t)$ is at a point on the circle of radius A at angle ωt to the x-axis. That is, $z(t)$ is going around the circle at a steady angular velocity ω. We can also write this:

$$z(t) = Ae^{i\omega t} = A\cos\omega t + iA\sin\omega t$$

and see that the point $z = x + iy$ is at coordinates $(x, y) = (A\cos\omega t, A\sin\omega t)$.



The angular velocity is ω, the actual velocity in the complex plane is $dz(t)/dt$.

Let's differentiate with respect to time:

$$\frac{d}{dt}Ae^{i\omega t} = \frac{d}{dt}A\left(\cos\omega t + i\sin\omega t\right) = i\omega Ae^{i\omega t} = i\omega A\left(\cos\omega t + i\sin\omega t\right) = i\omega A\cos\omega t - \omega A\sin\omega t.$$

Exercise: what are the x and y components of this velocity regarded as a vector? Show that it is perpendicular to the position vector. Why is that?

This differential equation has real and imaginary parts on both sides, so the real part on one side must be equal to the real part on the other side, and the same for imaginary parts. That gives

$$\frac{d}{dt}\cos\omega t = -\omega\sin\omega t, \quad \frac{d}{dt}\sin\omega t = \omega\cos\omega t$$

so differentiating the exponential is consistent with the standard results for trig functions.

Differentiating one more time,

$$\frac{d^2}{dt^2}Ae^{i\omega t} = -\omega^2 Ae^{i\omega t}$$

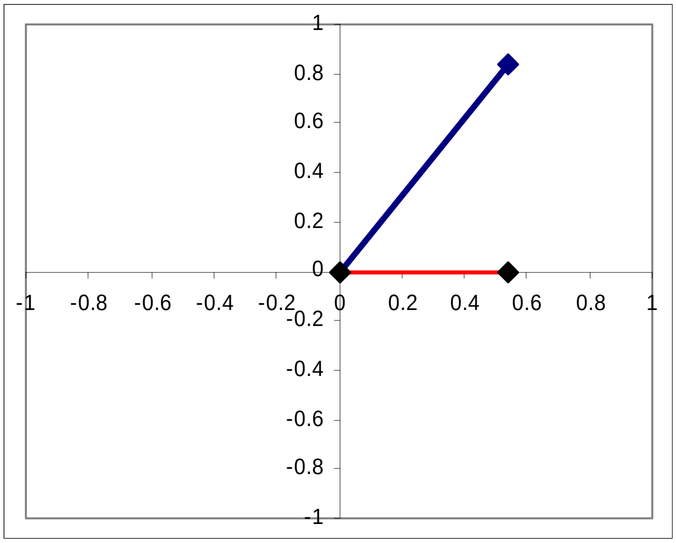
Again going to the picture of a complex numbers as a two-dimensional vector, this is just the acceleration of an object going round in a circle of radius A at angular velocity ω, and is just $A\omega^2$ towards the center of the circle, the familiar $r\omega^2 = v^2/r$. Thinking physics here, this is the motion of an object subject to a steady central force.

Follow the Shadow: Simple Harmonic Motion

But what if we just equate the real parts of both sides? That must be a perfectly good equation: it is

$$\frac{d^2}{dt^2}A\cos\omega t = -\omega^2 A\cos\omega t$$

This is just the x-component of the circling motion, that is, *it is the "shadow" of the circling point on the x-axis*:



A simple animation of this diagram can be found here.

Forgetting for the moment about the circling point, and staring at just this *x*-axis equation, we see it describes the motion of a point having acceleration towards the origin (that is, the minus sign ensures the acceleration is in the *opposite* direction to that of the point itself from the origin) and the magnitude of the acceleration is proportional to the distance of the point from the origin.

In fact, motion of this kind is very common in nature! It is called *simple harmonic motion*.

A simple standard example is a mass hanging on a spring. If it is initially at rest, and the string has length L (stretched from its natural length to balance mg) then if it is displaced a distance x from that equilibrium position, the spring will exert an extra force $-kx$ and the equation of motion will be

$$m\frac{d^2x}{dt^2} = -kx.$$

This is *exactly* the equation of motion satisfied by the "shadow" on the *x*-axis of a point circling at a steady rate.

The general solution is $x(t) = A\cos(\omega t + \delta)$, where a possible phase δ is included so that the point can be anywhere in its oscillation at $t = 0$.

Oscillations

Introduction

In this lecture, we will be looking at a wide variety of oscillatory phenomena. After a brief recap of undamped simple harmonic motion, we go on to look at a heavily damped oscillator. We do that before considering the lightly damped oscillator because the mathematics is a little more straightforward—for the heavily damped case, we don't need to use complex numbers. But they arise very naturally in the lightly damped case, and are great for understanding the *driven* oscillator and resonance phenomena, as will become apparent in later sections.

Brief Review of Undamped Simple Harmonic Motion

Our basic model simple harmonic oscillator is a mass m moving back and forth along a line on a smooth horizontal surface, connected to an inline horizontal spring, having spring constant k, the other end of the string being attached to a wall. The spring exerts a restoring force equal to $-kx$ on the mass when it is a distance x from the equilibrium point. By "equilibrium point" we mean the point corresponding to the spring resting at its natural length, and therefore exerting no force on the mass. The in-class realization of this model was an aircar, with a light spring above the track (actually, we used *two* light springs, going in opposite directions—we found if we just one it tended to sag on to the track when it was slack, but two in opposite directions could be kept taut. The two springs together act like a single spring having spring constant the sum of the two).

Newton's Law gives:

$$F = ma, \text{ or } m\frac{d^2x}{dt^2} = -kx.$$

Solving this differential equation gives the position of the mass (the aircar) relative to the rest position as a function of time:

$$x(t) = A\cos(\omega_0 t + \varphi).$$

Here A is the maximum displacement, and is called the **amplitude** of the motion. $\omega_0 t + \varphi$ is called the **phase**. φ is called the **phase constant**: it depends on where in the cycle you start, that is, where is the oscillator at time zero.

The velocity and acceleration are given by differentiating $x(t)$ once and twice:

$$v(t) = \frac{dx}{dt} = -A\omega_0 \sin(\omega_0 t + \varphi)$$

and

$$a(t) = \frac{d^2x(t)}{dt^2} = -A\omega_0^{\ 2}\cos(\omega_0 t + \varphi).$$

We see immediately that this $x(t)$ does indeed satisfy Newton's Law provided ω_0 is given by

$$\omega_0 = \sqrt{\frac{k}{m}}.$$

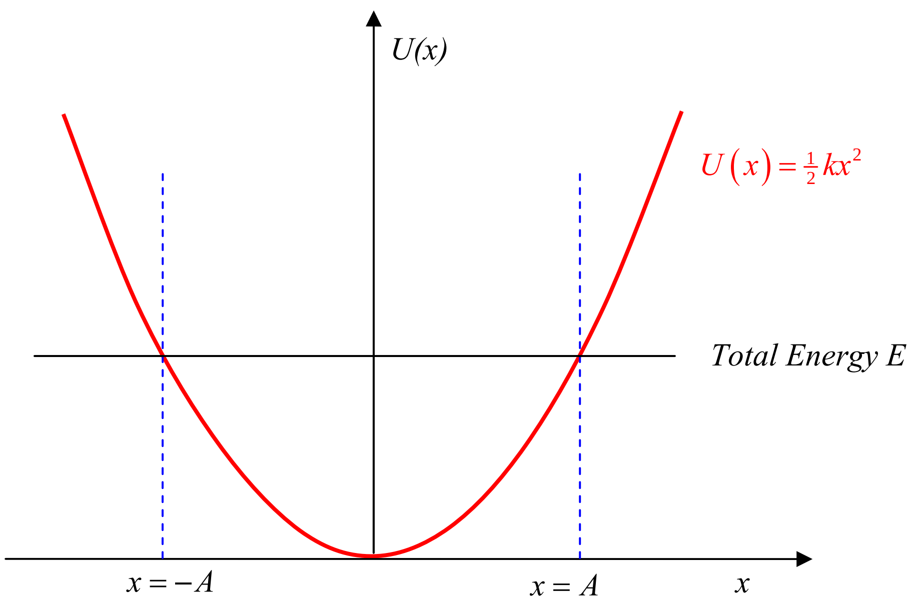
Exercise: Verify that, apart from a possible overall constant, this expression for ω_0 could have been figured out using dimensions.

Energy

The spring stores *potential energy*: if you push one end of the spring from some positive extension x to $x + dx$ (with the other end of the spring fixed, of course) the force $-kx$ opposes the motion, so you must push with force $+kx$, and therefore do work $kxdx$. To find the *total* potential energy stored by the spring when the end is x_0 away from the equilibrium point (natural length) we must find the total work required to stretch the spring from its natural length to an extension x_0. This means adding up all the little bits of work $kxdx$ needed to get the spring from no extension at all to an extension of x_0. In other words, we need to do an integral to find the potential energy $U(x_0)$:

$$U(x_0) = \int_0^{x_0} kxdx = \tfrac{1}{2}kx_0^{\,2}.$$

So the potential energy plotted as a function of distance from equilibrium is parabolic:



Potential Energy $U(x)$ for a Simple Harmonic Oscillator.
For ***total*** energy E, the oscillator swings back and forth between $x = -A$ and $x = +A$.

The oscillator has *total* energy equal to kinetic energy + potential energy,

$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2$$

when the mass is at position x. Putting in the values of $x(t)$, $v(t)$ from the equations above, it is easy to check that E is independent of time and equal to $\tfrac{1}{2}kA^2$, A being the amplitude of the motion, the maximum displacement. Of course, when the oscillator is at A, it is momentarily at rest, so has no kinetic energy.

A Heavily Damped Oscillator

Suppose now the motion is damped, with a drag force proportional to velocity. The equation of motion becomes:

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}.$$

Although this equation looks more difficult, it really isn't! The important point is that the terms are just *derivatives* of x with respect to time, multiplied by *constants*. It would be a lot more difficult if we had a drag force proportional to the square of the velocity, or if the force exerted by the spring were not a constant times x (this means we can't stretch the string too far!). Anyway, it is easy to find exponential functions that are solutions to this equation. Let us guess a solution:

$$x = x_0 e^{-\alpha t}.$$

Inserting this in the equation, using

$$\frac{dx}{dt} = -\alpha x_0 e^{-\alpha t}, \quad \frac{d^2x}{dt^2} = \alpha^2 x_0 e^{-\alpha t}$$

we find that it *is* a solution provided that α satisfies:

$$m\alpha^2 - \alpha b + k = 0$$

from which

$$\alpha = \frac{b \pm \sqrt{b^2 - 4mk}}{2m}.$$

Staring at this expression for α, we notice that for α to be real, we need to have

$$b^2 > 4mk.$$

What can that mean? Remember b is the damping parameter—we're finding that our proposed exponential solution only works for *large* damping! Let's analyze the large damping case now, then after that we'll go on to see how to extend the solution to small damping.

Interpreting the Two Different Exponential Solutions

It's worth looking at the case of *very* large damping, where the two exponential solutions turn out to decay at very different rates. For b^2 *much* greater than $4mk$, we can write

$$\alpha = \frac{b \pm b\sqrt{1 - \frac{4mk}{b^2}}}{2m} = \alpha_1, \alpha_2$$

and then expand the square root using

$$(1-x)^{1/2} \cong 1 - \tfrac{1}{2}x,$$

valid for small x, to find that approximately—for large b—the two possible values of α are:

$$\alpha_1 = \frac{b}{m} \text{ and } \alpha_2 = \frac{k}{b} \ .$$

That is to say, there are two possible highly damped decay modes,

$$x = A_1 e^{-\alpha_1 t} \text{ and } x = A_2 e^{-\alpha_2 t}.$$

Note that since the damping b is large, α_1 is *large*, meaning *fast* decay, and α_2 is *small*, meaning *slow* decay.

Question: what, physically, is going on in these two different highly damped exponential decays? Can you construct a plausible scenario of a mass on a spring, all in molasses, to see why two very different rates of change of speed are possible?

Hint: look again at the equation of motion of this damped oscillator. Notice that in each of these highly damped decays, one term doesn't play any part—but the irrelevant term is a *different* term for the two decays!

Answer 1: for $\alpha = k/b$, evidently the *mass* doesn't play a role. This decay is what you get if you pull the mass to one side, let go, then, after it gets moving, it will very slowly settle towards the equilibrium point. Its rate of approach is determined by balancing the spring's force against the speed-dependent damping force, to give the speed. The rate of change of speed—the acceleration—is so tiny that the *inertial* term—the mass—is negligible.

Answer 2: for $\alpha = b/m$, the *spring* is negligible. And, this is very *fast* motion ($b/m >> k/b$, since we said $b^2 >> 4mk$.) The way to get this motion is to pull the mass to one side, then give it a very strong kick towards the equilibrium point. If you give it just the right (high) speed, all the momentum you imparted will be spent overcoming the damping force as the mass moves to the center—the force of the spring will be negligible.

*The Most General Solution for the Highly Damped Oscillator

The damped oscillator equation

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$

is a linear equation. This means that if $x_1(t)$ is a solution, and $x_2(t)$ is another solution, that is,

$$m\frac{d^2x_1(t)}{dt^2} = -kx_1(t) - b\frac{dx_1(t)}{dt}$$
$$m\frac{d^2x_2(t)}{dt^2} = -kx_2(t) - b\frac{dx_2(t)}{dt}$$

then just adding the two equations we get:

$$m\frac{d^2(x_1(t)+x_2(t))}{dt^2} = -k(x_1(t)+x_2(t)) - b\frac{d(x_1(t)+x_2(t))}{dt}.$$

It is also clear that *multiplying a solution by a constant produces another solution*: if $x(t)$ satisfies the equation, so does $3x(t)$.

This means, then, that given two solutions $x_1(t)$ and $x_2(t)$, and two arbitrary constants A_1 and A_2, the function

$$A_1x_1(t) + A_2x_2(t)$$

is also a solution of the differential equation.

In fact, all possible motions of the highly damped oscillator have this form. The way to understand this is to realize that the oscillator's motion is *completely determined* if we specify at an initial instant of time *both* the position *and* the velocity of the oscillator. The equation of motion gives the acceleration as a function of position and velocity, so, at least in principle, we can work out step by step how the mass must move; technically, we are integrating the equation of motion, either mathematically, or numerically such as by using a spreadsheet. So, by suitably adjusting the two arbitrary constants A_1 and A_2, we can match our sum of solutions to any given initial position and velocity.

To summarize, for the highly damped oscillator any solution is of the form:

$$x(t) = A_1e^{-\alpha_1 t} + A_2e^{-\alpha_2 t} = A_1e^{-\frac{b+b\sqrt{1-\frac{4mk}{b^2}}}{2m}t} + A_2e^{-\frac{b-b\sqrt{1-\frac{4mk}{b^2}}}{2m}t}.$$

Exercises on highly damped oscillations

1. If the oscillator is pulled aside a distance x_0, and released from rest at $t = 0$, what are A_1, A_2? Describe the subsequent motion, especially the very beginning: what is the initial acceleration? (*Hint*: think carefully about how important the damping term is immediately after release from rest—you should be able to *guess* the initial acceleration.)

2. If the oscillator is initially at the equilibrium position $x_0 = 0$, but is given a kick to a velocity v_0, find A_1 and A_2 and describe the subsequent motion.

*The Principle of Superposition for Linear Differential Equations

The equation for the highly damped oscillator is a linear differential equation, that is, an equation of the form (in more usual notation):

$$c_0 f(x) + c_1 \frac{df(x)}{dx} + c_2 \frac{d^2 f(x)}{dx^2} = 0$$

where c_0, c_1 and c_2 are constants, that is, independent of x.

For such a linear differential equation, if $f_1(x)$ and $f_2(x)$ are solutions, so is $A_1 f_1(x) + A_2 f_2(x)$ for any constants A_1, A_2. This is called the ***Principle of Superposition***, and is proved in general *exactly* as we proved it for the highly damped oscillator in the preceding section.

Even more important, this Principle of Superposition is valid, using analogous arguments, for linear differential equations in *more than one variable*, such as the wave equations we shall be considering shortly. In that case, it gives insight into how waves can pass through each other and emerge unchanged.

A Lightly Damped Oscillator

We can go through exactly the same mathematical steps in solving the equation of motion as we did for the heavily damped case: we look for solutions of the form

$$x = x_0 e^{-\alpha t}$$

and as before we find there are solutions with

$$\alpha_1, \alpha_2 = \frac{b \pm \sqrt{b^2 - 4mk}}{2m}.$$

But the difference is that for *light* damping, by which we mean $b^2 < 4mk$, the expression inside the square root is negative! We are going to have to work with the square root of a negative number. We do this formally by writing:

$$\sqrt{b^2 - 4mk} = i\sqrt{4mk - b^2}$$

with $i^2 = -1$ as usual. This gives the two possible exponential solutions:

$$x_1(t) = e^{-\frac{bt}{2m}} e^{-\frac{i\sqrt{4mk-b^2}}{2m}t}, \quad x_2(t) = e^{-\frac{bt}{2m}} e^{+\frac{i\sqrt{4mk-b^2}}{2m}t}.$$

and a general solution

$$x(t) = A_1 e^{-\frac{bt}{2m}} e^{-\frac{i\sqrt{4mk-b^2}}{2m}t} + A_2 e^{-\frac{bt}{2m}} e^{+\frac{i\sqrt{4mk-b^2}}{2m}t}.$$

Of course, the position of the mass $x(t)$ has to be a real number! We must choose A_1 and A_2 to make sure this is so. If we choose

$$A_1 = \tfrac{1}{2} A e^{-i\delta}, \quad A_2 = \tfrac{1}{2} A e^{+i\delta}$$

where A and δ are real, and remembering

$$\cos\theta = \tfrac{1}{2}(e^{+i\theta} + e^{-i\theta}),$$

we find

$$x(t) = Ae^{-\frac{bt}{2m}} \cos\left(\frac{\sqrt{4mk-b^2}}{2m} t + \delta \right).$$

This is the most general real solution of the lightly damped oscillator—the two arbitrary constants are the amplitude A and the phase δ. So for small b, we get a cosine oscillation multiplied by a gradually decreasing function, $e^{-bt/2m}$.

This is often written in terms of a **decay time** τ defined by

$$\tau = m/b.$$

The amplitude of oscillation A therefore decays in time as $e^{-t/2\tau}$, and the *energy* of the oscillator (proportional to A^2) decays as $e^{-t/\tau}$. This means that in time τ the energy is down by a factor $1/e$, with $e = 2.71828\ldots$

The solution is sometimes written

$$x(t) = Ae^{-\frac{bt}{2m}} \cos(\omega' t + \delta)$$

where

$$\omega'^2 = \frac{4mk - b^2}{4m^2} = \frac{k}{m} - \frac{b^2}{4m^2} = \omega_0^2 - \frac{b^2}{4m^2}.$$

Notice that for small damping, the oscillation frequency doesn't change much from the undamped value: the change is proportional to the *square* of the damping.

The *Q* Factor

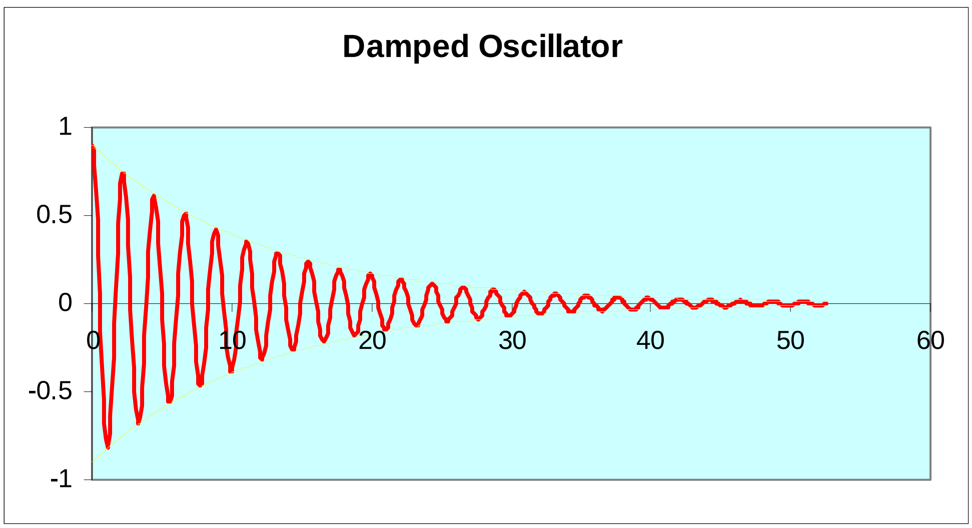
The Q factor is a measure of the "quality" of an oscillator (such as a bell): how long will it keep ringing once you hit it? Essentially, it is a measure of how many oscillations take place during the time the energy decays by the factor of $1/e$.

Q is defined by:

$$Q = \omega_0 \tau$$

so, strictly speaking, it measures how many radians the oscillator goes around in time τ. For a typical bell, τ would be a few seconds, if the note is middle C, 256 Hz, that's $\omega_0 = 2\pi \times 256$, so Q would be of order a few thousand.

Exercise: estimate Q for the following oscillator (and don't forget the energy is proportional to the *square* of the amplitude):



The yellow curves in the graph above are the pair of functions $+e^{-bt/2m}$, $-e^{-bt/2m}$, often referred to as the *envelope* of the oscillation curve, as they "envelope" it from above and below.

*Critical Damping

There is just one case we haven't really discussed, and it's called "critical damping": what happens when $b^2 - 4mk$ is exactly zero? At first glance, that sounds easy to answer: there's just the one solution

$$x(t) = Ae^{-\frac{bt}{2m}}.$$

But that's not good enough—it tells us that if we begin at $t = 0$ with the mass at x_0, it must have velocity dx/dt equal to $-x_0b/2m$. But, in fact, we can put the mass at x_0 and kick it to any initial velocity we want! So what happened to the other solution?

We can get a clue by examining the two exponentially falling solutions for the overdamped case as we approach critical damping:

$$x(t) = A_1 e^{-\frac{b+b\sqrt{1-\frac{4mk}{b^2}}}{2m}t} + A_2 e^{-\frac{b-b\sqrt{1-\frac{4mk}{b^2}}}{2m}t}$$

As we approach critical damping, the small quantity

$$\varepsilon = \frac{\sqrt{b^2 - 4mk}}{2m}$$

approaches zero. The general solution to the equation has the form

$$x(t) = e^{-\frac{bt}{2m}}(A_1 e^{-\varepsilon t} + A_2 e^{+\varepsilon t}).$$

This is a valid solution for any real A_1, A_2. To find the solution we're missing, the trick is to take $A_2 = -A_1$. In the limit of small ε, we can take $e^{\varepsilon t} = 1 + \varepsilon t$, and we discover the solution

$$x(t) = -e^{-\frac{bt}{2m}} 2\varepsilon t.$$

As usual, we can always multiply a solution of a linear differential equation by a constant and still have a solution, so we write our new solution as

$$x(t) = A_2 t e^{-\frac{bt}{2m}}.$$

The general solution to the critically damped oscillator then has the form:

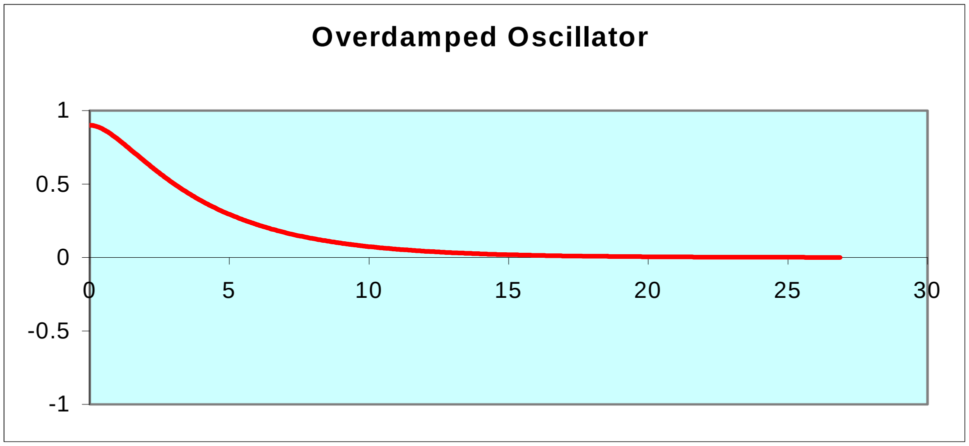
$$x(t) = (A_1 + A_2 t)e^{-\frac{bt}{2m}}.$$

Exercise: check that this *is* a solution for the critical damping case, and verify that solutions of the form t times an exponential *don't* work for the other (*non*critical damping) cases.

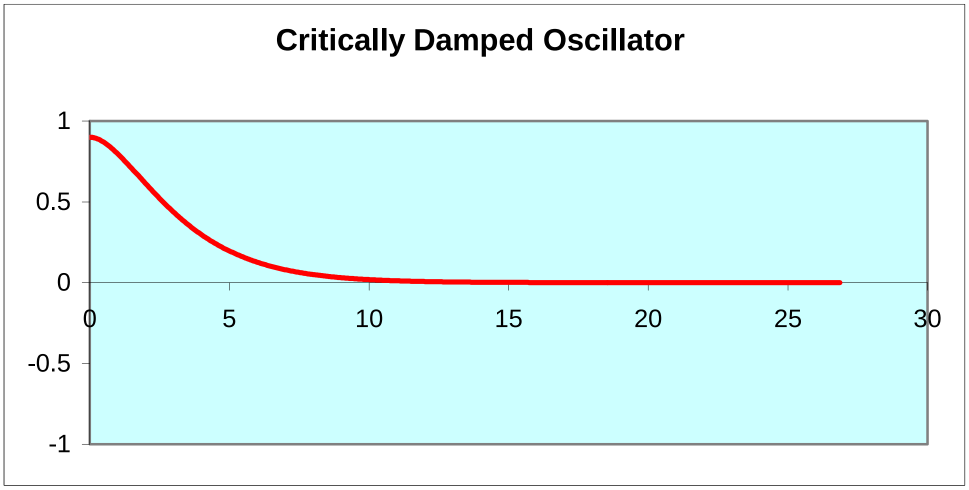
Shock Absorbers and Critical Damping

A shock absorber is basically a damped spring oscillator, the damping is from a piston moving in a cylinder filled with oil. Obviously, if the oil is very thin, there won't be much damping, a pothole will cause your car to bounce up and down a few times, and shake you up. On the other hand, if the oil is *really* thick, or the piston too tight, the shock absorber will be too stiff—it won't absorb the shock, and you will! So we need to tune the damping so that the car responds smoothly to a bump in the road, but doesn't continue to bounce after the bump.

Clearly, the "Damped Oscillator" graph in the Q-factor section above corresponds to too little damping for comfort from a shock absorber point of view, such an oscillator is said to be **underdamped**. The opposite case**, overdamping**, looks like this:

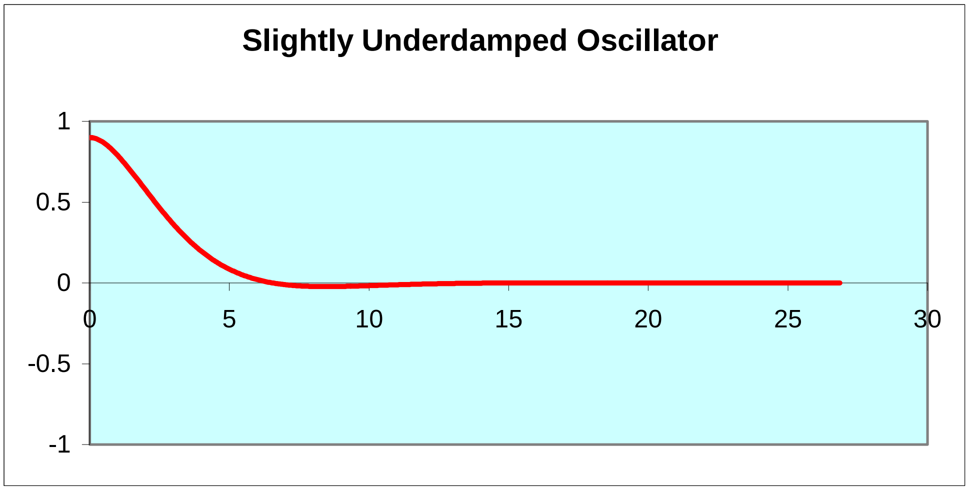


The dividing line between overdamping and underdamping is called **critical damping**. Keeping everything constant except the damping force from the graph above, critical damping looks like:



This corresponds to $\omega' = 0$ in the equation for $x(t)$ above, so it is a purely exponential curve. Notice that the oscillator moves more quickly to zero than in the overdamped (stiff oil) case.

You might think that critical damping is the best solution for a shock absorber, but actually a little less damping might give a better ride: there would be a slight amount of bouncing, but a quicker response, like this:



You can find out how your shock absorbers behave by pressing down one corner of the car and then letting go. If the car clearly bounces around, the damping is too little, and you need new shocks.

A Driven Damped Oscillator: the Equation of Motion

We are now ready to examine a very important case: the driven damped oscillator. By this, we mean a damped oscillator as analyzed above, but with a periodic external force driving it. If the driving force has the same period as the oscillator, the amplitude can increase, perhaps to disastrous proportions, as in the famous case of the Tacoma Narrows Bridge.

The equation of motion for the driven damped oscillator is:

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=F_0\cos\omega t.$$

We shall be using ω for the frequency of the driving force, and ω_0 for the natural frequency of the oscillator if the damping term is ignored, $\omega_0=\sqrt{k/m}$.

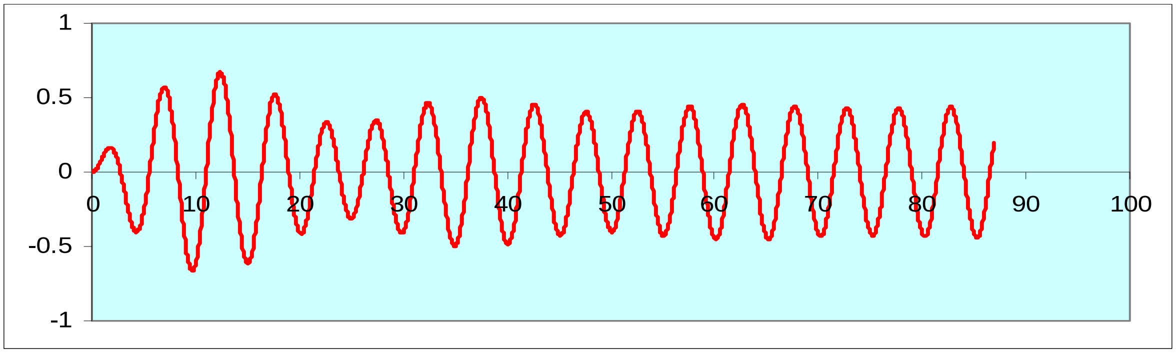
The Steady State Solution and Initial Transient Behavior

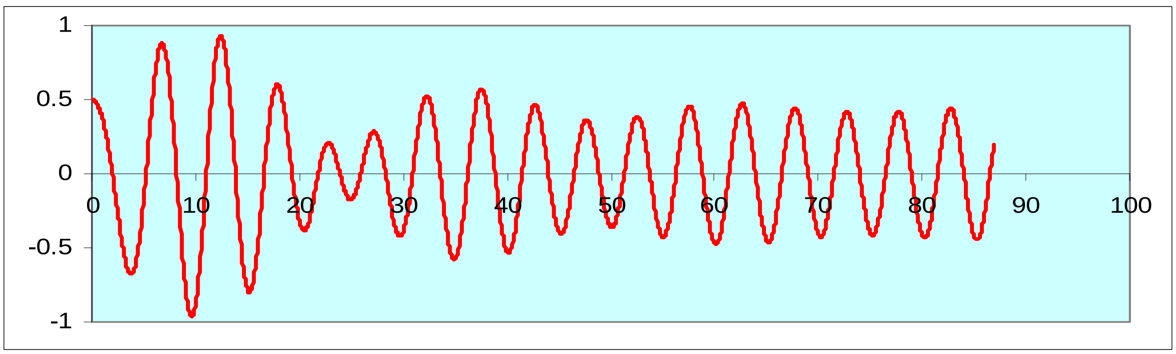
The solution to this differential equation is not unique: as with any second order differential equation, there are two constants of integration, which are determined by specifying the initial position and velocity.

However, as we shall prove below using complex numbers, the equation *does* have a *unique* steady state solution with x oscillating at the same frequency as the external drive. How can that be fitted to arbitrary initial conditions? The key is that we can add to the steady state solution any solution of the *undriven* equation $m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0,$ and we'll clearly still have a solution of the full damped driven equation. We know what those undriven solutions look like:

they all die away as time goes on. So, we can add such a solution to fit the specified initial conditions, and after a while the system will lose memory of those conditions and settle into the steady driven solution. The initial deviations from the steady solution needed to satisfy initial conditions are termed *transients*.

Here's a pair of examples: the same driven damped oscillator, started with zero velocity, once from the origin and once from 0.5:





Notice that after about 70 seconds, the two curves are the same, both in amplitude and phase.

Using Complex Numbers to Solve the Steady State Equation Easily

We begin by writing:

$$\text{external driving force} = F_0 e^{i\omega t}$$

with F_0 real, so the *real* driving force is just the real part of this, $F_0 \cos \omega t$.
So now we're trying to solve the equation

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 e^{i\omega t}.$$

We'll try the complex function, $x(t) = Ae^{i(\omega t+\varphi)}$, with A a real number, $x(t)$ cycling at the same rate as the driving force. We can always take the amplitude A to be real: that is *not* a restriction, since we've added the adjustable phase factor $e^{i\varphi}$. Physically, this factor allows the solution to lag the driver in phase, as indeed we shall find to be the case. If we succeed in finding an $x(t)$ that satisfies the equation, the real parts of the two sides of the equation must be equal:

If $x(t) = Ae^{i(\omega t+\varphi)}$ is a solution to the equation with the complex driving force, $F_0 e^{i\omega t}$, its real part, $A\cos(\omega t+\varphi)$, will be a solution to the equation with the real driving force, $F_0\cos\omega t$.

It's very easy to check that $x(t) = Ae^{i(\omega t+\varphi)}$ *is* a solution to the equation, with the right A and φ! Just put it in and see what happens. The differentiations are simple, giving

$$-m\omega^2 Ae^{i(\omega t+\varphi)} + ib\omega Ae^{i(\omega t+\varphi)} + kAe^{i(\omega t+\varphi)} = F_0 e^{i\omega t}.$$

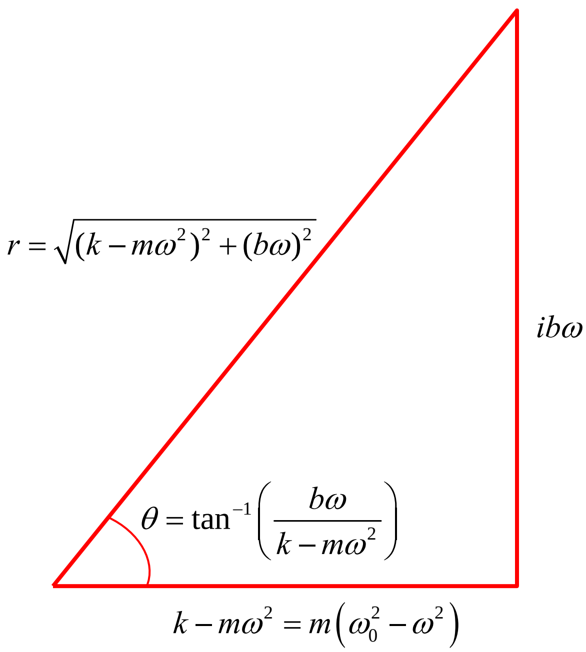
To nail down A and φ, we begin by cancelling out the common factor $e^{i\omega t}$, then shifting the $e^{i\varphi}$ to the other side, to find

$$A = \frac{F_0 e^{-i\varphi}}{k - m\omega^2 + ib\omega}$$

To get some insight into this equation, let us diagram that complex number $k - m\omega^2 + ib\omega$.

It has real part $k - m\omega^2$ and imaginary part $ib\omega$.

Its phase is the angle θ: that is, $k - m\omega^2 + ib\omega = re^{i\theta}$.



The complex number $k - m\omega^2 + ib\omega$

Putting this in the equation, we have

$$A = \frac{F_0 e^{-i\varphi}}{k - m\omega^2 + ib\omega} = \frac{F_0 e^{-i\varphi}}{re^{i\theta}} = \frac{F_0}{r} e^{-i(\varphi+\theta)}$$

and since A, F_0 and r are real, $e^{-i(\varphi+\theta)}$ must be real as well: so $\varphi = -\theta$, and we see that the amplitude A of the oscillations is given by

$$A = \frac{F_0}{r} = \frac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2+(b\omega)^2}}, \quad x(t) = Ae^{i(\omega t-\theta)},$$

where we've written $k = m\omega_0^2$.

So we've already solved the differential equation: the amplitude A is proportional to the strength of the driving force, and that ratio is determined by the parameters of the undriven oscillator.

The important thing to note about the amplitude A is that if the damping b is small, A gets *very large* when the frequency of the driver approaches the natural frequency of the oscillator! This is called *resonance*, and is what happened to the Tacoma Narrows Bridge. Of course, it has its positive aspects, from getting a swing going to tuning a radio.

The *phase lag* of the oscillations behind the driver, $\theta = \tan^{-1}\left(b\omega/\left(k-m\omega^2\right)\right)$, is completely determined by the frequency together with the physical constants of the undriven oscillator: the mass, spring constant, and damping strength. So, when the driving force $F_0e^{i\omega t}$ generates the motion $x(t) = Ae^{i(\omega t+\varphi)} = Ae^{i(\omega t-\theta)}$, the lag angle θ is independent of the strength of the driving force: a stronger force doesn't get the oscillator more in sync, it just increases the amplitude of the oscillations.

Note that at *low* frequencies, $\omega \ll \omega_0$, the oscillator lags behind by a small angle, but at resonance $\omega = \omega_0$ $\theta = \pi/2$, and for driving frequencies above ω_0, $\theta > \pi/2$.

Back to Reality

To summarize: we've just established that $x(t) = Ae^{i(\omega t-\theta)}$ with $A = F_0/\sqrt{m^2(\omega_0^2-\omega^2)^2+(b\omega)^2}$ and $\theta = \tan^{-1}\left(b\omega/\left(k-m\omega^2\right)\right)$ is a solution to the driven damped oscillator equation $m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx = F_0e^{i\omega t}$ with the complex driving force $F_0e^{i\omega t}$.

So, *equating the real parts of the two sides of the equation*, since *m*, *b*, *k* are all real,

$$x = A\cos(\omega t-\theta)$$

is a solution of the equation with the <u>real</u> driving force $F_0\cos\omega t$ **.**

We could have found this out without complex numbers, by using a trial solution $A\cos(\omega t+\varphi)$. However, it's not that easy—the left hand side becomes a mix of sines and cosines, and one needs to use trig identities to sort it all out. With a little practice, the complex method is easier and is certainly more direct.

Now the total energy of the oscillator is

$$\begin{aligned} E &= \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 \\ &= \tfrac{1}{2}mv^2 + \tfrac{1}{2}m\omega_0^2x^2. \end{aligned}$$

Putting in

$$x(t) = A\cos(\omega t - \theta), \quad v(t) = -A\omega\sin(\omega t - \theta)$$

gives

$$E = \tfrac{1}{2}mA^2\left(\omega^2\sin^2(\omega t - \theta) + \omega_0^2\cos^2(\omega t - \theta)\right).$$

Note that this is *not* constant through the cycle unless the oscillator is at resonance, $\omega = \omega_0$.

We can see from the above that at the resonant frequency, $E = \tfrac{1}{2}m\omega_0^2A^2$, and from the section above

$$A = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + (b\omega)^2}},$$

so the energy in the oscillator at the resonant frequency is

$$E_{\text{resonance}} = \tfrac{1}{2}m\omega_0^2A^2 = \tfrac{1}{2}m\omega_0^2\frac{F_0^2}{b^2\omega_0^2} = \tfrac{1}{2}m\frac{F_0^2}{b^2} = \frac{Q^2}{2}\frac{F_0^2}{m\omega_0^2},$$

recalling that $Q = \omega_0\tau = \omega_0 m/b$.

So Q, the quality factor, the measure of how long an oscillator keeps ringing, also measures the strength of response of the oscillator to an external driver at the resonant frequency.

But what happens on going *away* from the resonant frequency? Let's assume that Q is large, and the driving force is kept constant. It won't take much change in ω from ω_0 for the denominator $m^2(\omega_0^2 - \omega^2)^2 + (b\omega)^2$ in the expression for E to double in size. In fact, for large Q, it's a good approximation to replace $b\omega$ by $b\omega_0$ over that variation, and it is then straightforward to check that the energy in the oscillator drops to one-half its resonant value for $\omega - \omega_0 \cong \pm\omega_0/2Q$.

Exercise: prove this.

The bottom line is that for increasing Q, the response at the resonant frequency gets larger, but this large response takes place over a narrower and narrower range in driving frequencies.

And Now to Work…

An important practical question is: how much *work* is the driver doing to keep this thing going?

It's simplest to work with the real solution. Suppose the oscillator moves through Δx in a time Δt, the driving force does work $\left(F_0 \cos \omega t\right)\Delta x$, so

$$\text{rate of working at time } t \;=\; \left(F_0 \cos \omega t\right)\left(\Delta x / \Delta t\right) = \left(F_0 \cos \omega t\right) v\left(t\right)$$

The important thing is the *average* rate of working of the driving force, the *mean power input*, found by averaging over a complete cycle:

From $x(t) = A\cos(\omega t - \theta)$, $v\left(t\right) = -A\omega \sin\left(\omega t - \theta\right)$, averaging the power input (the bar above means average over a complete cycle) and denoting average power by *P*,

$$\begin{aligned}
P &= F_0 \overline{\left(\cos \omega t\right) v\left(t\right)} \\
&= -F_0 A\omega \overline{\cos \omega t \sin\left(\omega t - \theta\right)} \\
&= -F_0 A\omega \overline{\cos \omega t \sin \omega t} \cos\theta + F_0 A\omega \overline{\cos^2 \omega t} \sin\theta \\
&= \tfrac{1}{2} F_0 A\omega \sin\theta
\end{aligned}$$

since over one cycle the average $\overline{\cos^2 \omega t} = \tfrac{1}{2}$ and $\overline{\cos \omega t \sin \omega t} = \tfrac{1}{2}\overline{\sin 2\omega t} = 0$ (Remembering $\cos^2 \omega t + \sin^2 \omega t = 1$ at all times, and sine is just cosine moved over, so they must have the same average over a complete cycle.)

This can be expressed entirely in terms of the driving force and frequency:

Since

$$A = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + (b\omega)^2}}, \quad \sin\theta = \frac{b\omega}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + (b\omega)^2}}$$

$$\begin{aligned}
P &= \tfrac{1}{2} F_0 A\omega \sin\theta \\
&= \frac{1}{2} \frac{b\omega^2 F_0^2}{m^2(\omega_0^2 - \omega^2)^2 + (b\omega)^2}
\end{aligned}$$

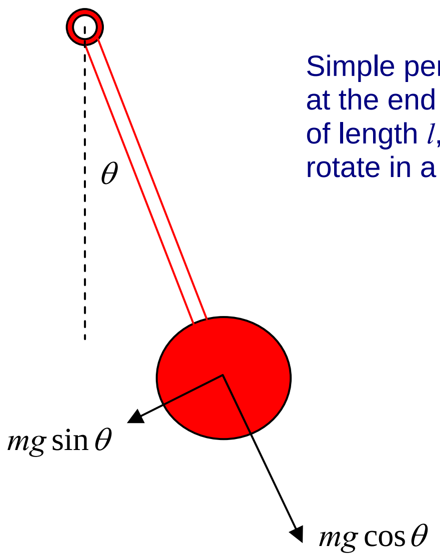
Exercise 1: Prove that for a lightly damped oscillator, at resonance the oscillator extracts the most work from the driving force.

Exercise 2: Prove that any solution of the damped oscillator equation (with $F = 0$) can be added to the driven oscillator solution, and gives another solution to the driven oscillator. How do you pick the "right solution"?

The Pendulum

The Simple Pendulum

Galileo was the first to record that the period of a swinging lamp high in a cathedral was independent of the amplitude of the oscillations, at least for the small amplitudes he could observe. In 1657, Huygens constructed the first pendulum clock, a vast improvement in timekeeping over all previous techniques. So the pendulum was the first oscillator of real technological importance.



Simple pendulum: a mass *m* at the end of a rigid light rod of length *l*, constrained to rotate in a vertical plane.

In fact, though, the pendulum is not quite a simple harmonic oscillator: the period *does* depend on the amplitude, but provided the angular amplitude is kept small, this is a small effect.

The weight *mg* of the bob (the mass at the end of the light rod) can be written in terms of components parallel and perpendicular to the rod. The component parallel to the rod balances the tension in the rod. The component perpendicular to the rod accelerates the bob,

$$ml\frac{d^2\theta}{dt^2} = -mg\sin\theta.$$

The mass cancels between the two sides, pendulums of different masses having the same length behave identically. (In fact, this was one of the first tests that inertial mass and gravitational mass are indeed equal: pendulums made of different materials, but the same length, had the same period.)

For small angles, the equation is close to that for a simple harmonic oscillator,

$$l\frac{d^2\theta}{dt^2} = -g\theta,$$

with frequency $\omega = \sqrt{g/l}$, that is, time of one oscillation $T = 2\pi\sqrt{l/g}$. At a displacement of ten degrees, the simple harmonic approximation overestimates the restoring force by around one part in a thousand, and for smaller angles this error goes essentially as the cube of the angle. So a pendulum clock designed to keep time with small oscillations of the pendulum will gain four seconds an hour or so if the pendulum is made to swing with a maximum angular displacement of ten degrees.

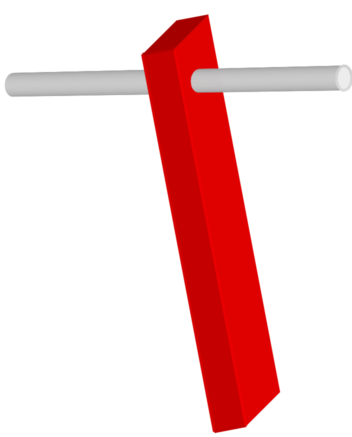
The potential energy of the pendulum relative to its rest position is just *mgh*, where *h* is the height difference, that is, $mgl(1-\cos\theta)$. The total energy is therefore

$$E = \tfrac{1}{2}m\left(l\frac{d\theta}{dt}\right)^2 + mgl(1-\cos\theta) \cong \tfrac{1}{2}m\left(l\frac{d\theta}{dt}\right)^2 + \tfrac{1}{2}mgl\theta^2$$

for small angles.

Pendulums of Arbitrary Shape

The analysis of pendulum motion in terms of angular displacement works for any rigid body swinging back and forth about a horizontal axis under gravity. For example, consider a rigid rod.



The kinetic energy is given by $\tfrac{1}{2}I\dot{\theta}^2$, where *I* is the moment of inertia of the body about the rod, the potential energy is $mgl(1-\cos\theta)$ as before, *but l* is now the *distance of the center of mass from the axis*.

The equation of motion is that the rate of change of angular momentum equals the applied torque,

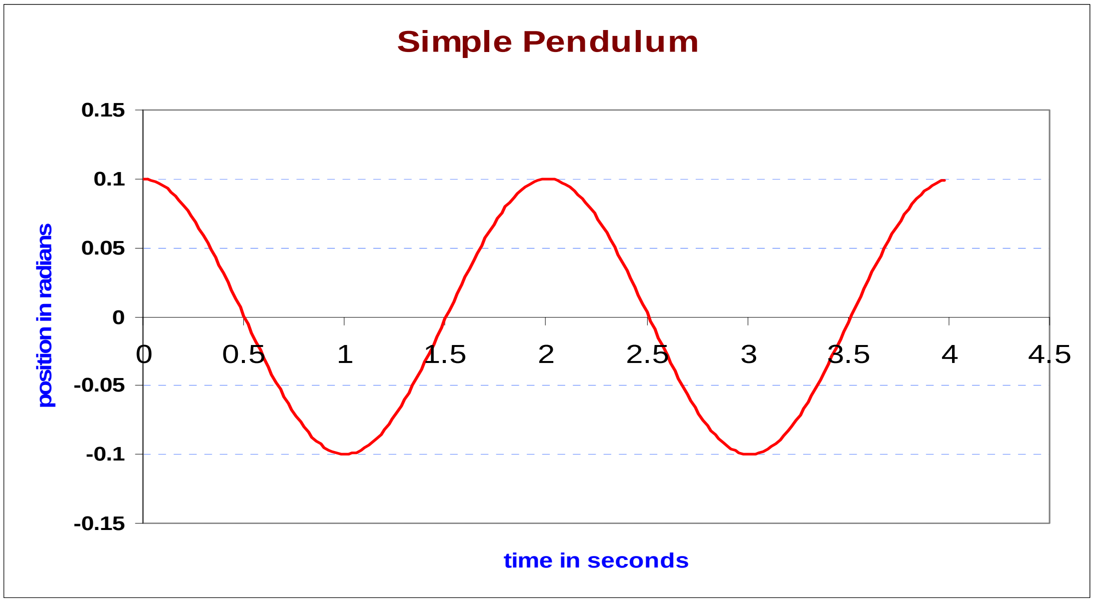
$$I\ddot{\theta} = -mgl\sin\theta,$$

for small angles the period $T = 2\pi\sqrt{I/mgl}$, and for the simple pendulum we considered first $I = ml^2$, giving the previous result.

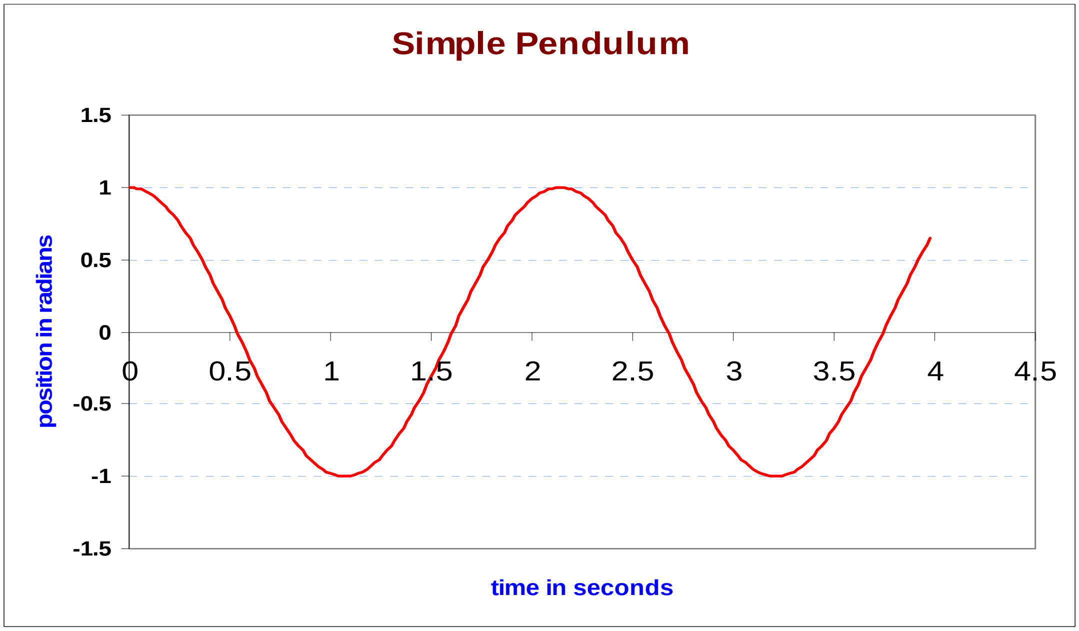
Variation of Period of a Pendulum with Amplitude

As the amplitude of pendulum motion increases, the period lengthens, because the restoring force $-mg\sin\theta$ increases more slowly than $-mg\theta$ ($\sin\theta \cong \theta - \theta^3/3!$ for small angles). The simplest way to get some idea how this happens is to explore it with the accompanying spreadsheet.

Begin with an initial displacement of 0.1 radians (5.7 degrees):



Next, try one radian:



The change in period is a little less that 10%, not too dramatic considering the large amplitude of this swing.

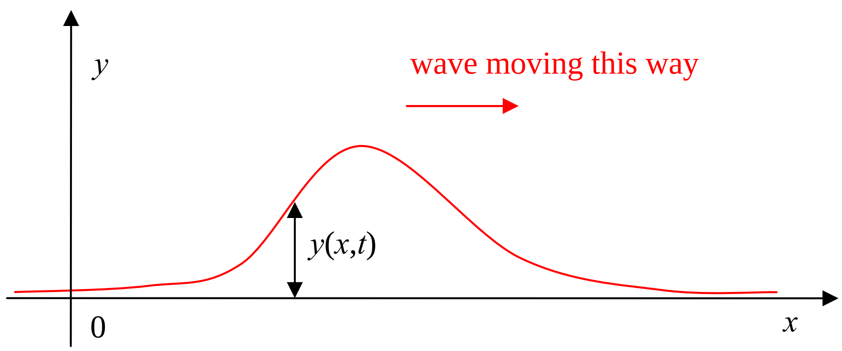
Two radians gives an increase around 35%, and three radians amplitude increases the period almost threefold. It's well worth exploring further with the spreadsheet.

Introducing Waves: Strings and Springs

One-Dimensional Traveling Waves

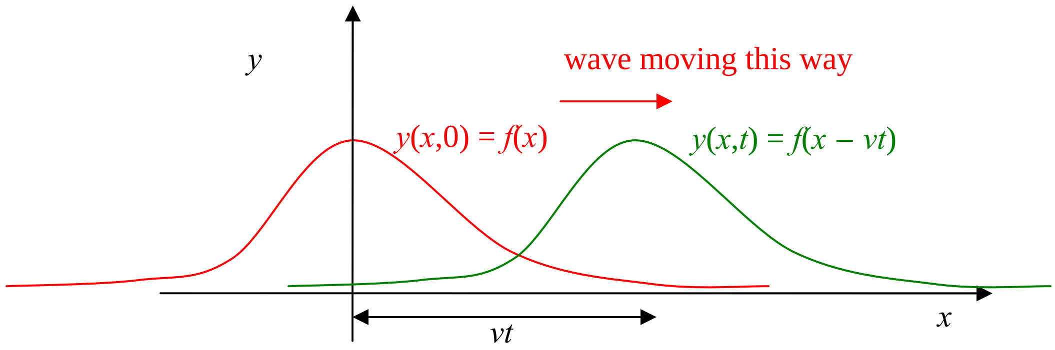
The most important kinds of traveling waves in everyday life are electromagnetic waves, sound waves, and perhaps water waves, depending on where you live. (Electromagnetic waves include X-rays, light, heat, microwaves, radio, etc.) But it's tough to analyze waves spreading out in three dimensions, reflecting off objects, etc., so we begin with the simplest interesting examples of waves, those restricted to move along a line.

Let's start with a rope, like a clothesline, stretched between two hooks. You take one end off the hook, holding the rope, and, keeping it stretched fairly tight, wave your hand up and back once. If you do it fast enough, you'll see a single bump travel along the rope:



This is the simplest example of a ***traveling wave***. You can make waves of different shapes by moving your hand up and down in different patterns, for example an upward bump followed by a dip, or two bumps. You'll find that the traveling wave *keeps the same shape* as it moves down the rope. (That's before it reaches the end, of course—things get more complicated at that point—we'll discuss it later.)

Taking the rope to be stretched tightly enough that we can take it to be horizontal, we'll use its rest position as our x-axis (see the diagram above). The y-axis is taken vertically upwards, and we only wave the rope in an up-and-down way, so actually $y(x,t)$ will be how far the rope is from its rest position at x at time t: that is, the graph $y(x,t)$ above just shows *where the rope is* at time t.



We can now express the observation that the wave "keeps the same shape" more precisely. Taking for convenience time $t = 0$ to be the moment when the peak of the wave passes $x = 0$, we graph here the rope's position at $t = 0$ (red) and some later time t (green). Denoting the first function by $y(x,0) = f(x)$, then the second $y(x,t) = f(x - vt)$: it's *the same function*—the "same shape"—but moved over by vt, where v is the velocity of the wave.

To summarize: on sending a traveling wave down a rope by jerking the end up and down, from observation the wave travels at constant speed and keeps its shape, so the displacement y of the rope at any horizontal position at x at time t has the form

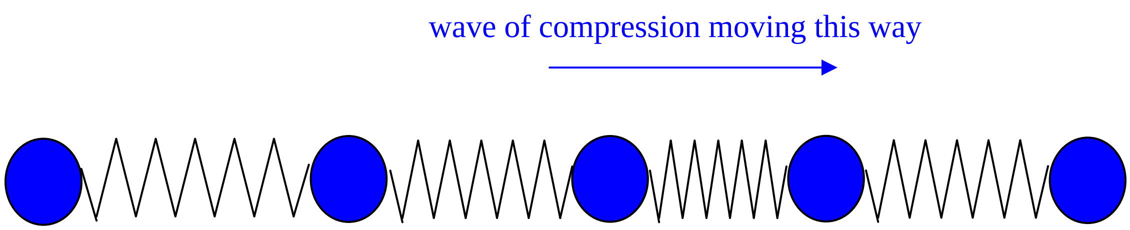
$$y(x,t) = f(x - vt).$$

(We're neglecting frictional effects—in a real rope, the bump gradually gets smaller as it moves along.)

Transverse and Longitudinal Waves

The wave on a rope described above is called a ***transverse wave***, because, as the wave passes, the motion of any actual bit of rope is in the y-direction, at right angles (transverse) to the direction of the wave itself, which is of course along the rope.

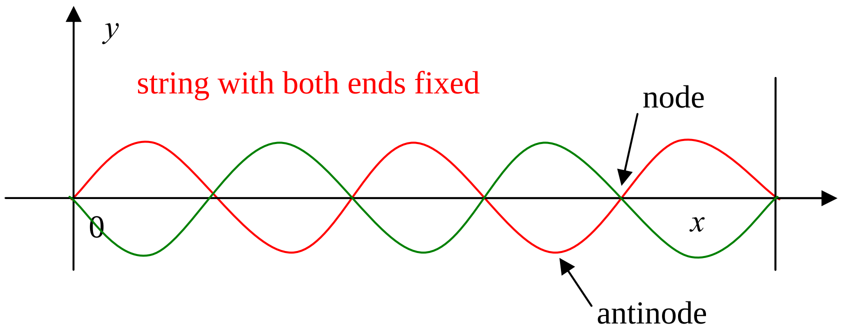
A different kind of wave is possible: consider a series of balls in a line connected by springs, and give the ball on the far left a sudden push to the right. A wave of compression will move down the line:



In this case, the motion of each ball as the wave passes through is *in the same direction as the wave*. In fact, this happens as a sound wave travels through air: it's a ***longitudinal wave***.

Traveling and Standing Waves

Both the waves considered above are *traveling* waves. Another familiar kind of wave is that generated on a string fixed at both ends when it is made to vibrate. We found in class that for certain frequencies the string vibrated in a sine-wave pattern, as illustrated below, with no vibration at the ends, of course, but also no vibration at a series of equally-spaced points between the ends: these quiet places we term ***nodes***. The places of maximum oscillation are ***antinodes***. We found a sequence of these standing waves on increasing the driving frequency, having 0, 1, 2, 3, … nodes. The red and green curves indicate the string position at successive times.

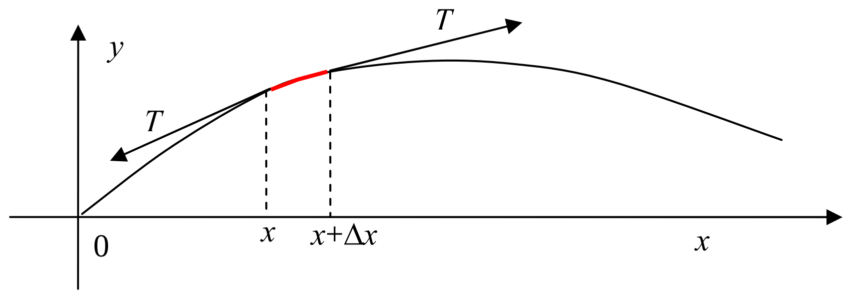


Almost all musical instruments generate standing waves: the piano has standing waves on strings, the organ generates standing waves in the air in pipes. Other instruments are more complicated: although the sound of a violin comes from a vibrating string, resonance with the rest of the instrument gives rise to complicated standing wave patterns. An excellent discussion and demonstration can be found at http://www.phys.unsw.edu.au/music/violin/ , along with links to similar pages for other instruments, and many aspects of sound and music.

Analyzing Waves on a String

From Newton's Laws to the Wave Equation

Everything there is to know about waves on a uniform string can be found by applying Newton's Second Law, $\vec{F} = m\vec{a}$, to one tiny bit of the string. Well, at least this is true of the small amplitude waves we shall be studying—we'll be assuming the deviation of the string from its rest position is small compared with the wavelength of the waves being studied. This makes the math simpler, and is an excellent approximation for musical instruments, etc. Having said that, we'll draw diagrams, like the one below, with rather large amplitude waves, to show more clearly what's going on.

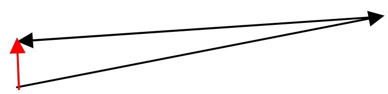


Let's write down $\vec{F} = m\vec{a}$ for the small length of string between x and $x + \Delta x$ in the diagram above.

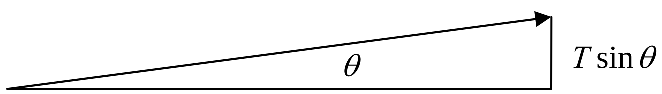
Taking the string to have mass density μ kg/m, we have $m = \mu \Delta x$.

The forces on the bit of string (neglecting the tiny force of gravity, air resistance, etc.) are the tensions T at the two ends. The tension will be uniform in magnitude along the string, but the string curves if it's waving, so the two $\vec{T}$ vectors at opposite ends of the bit of string do not quite cancel, this is the net force $\vec{F}$ we're looking for.

Bearing in mind that we're only interested here in *small amplitude* waves, we can see from the diagram (squashing it mentally in the y-direction) that both $\vec{T}$ vectors will be close to horizontal, and, since they're pointing in opposite directions, their sum—the net force $\vec{F}$—will be very close to vertical:



The vertical component of the tension $\vec{T}$ at the $x + \Delta x$ end of the bit of string is $T \sin\theta$, where θ is the angle of slope of the string at that end. This slope is of course just $dy(x+\Delta x)/dx$, or, more precisely, $dy/dx = \tan\theta$.



However, if the wave amplitude is small, as we're assuming, then θ is small, and we can take $\tan\theta = \sin\theta = \theta$, and therefore take the vertical component of the tension force on the string to be $T\theta = T dy(x+\Delta x)/dx$. So the total vertical force from the tensions at the two ends becomes

$$\vec{F} = T\left(\frac{dy(x+\Delta x)}{dx} - \frac{dy(x)}{dx}\right) \cong T\frac{d^2 y(x)}{dx^2}\Delta x$$

the equality becoming exact in the limit $\Delta x \to 0$.

At this point, it is necessary to make clear that *y* is a function of *t* as well as of *x*: $y = y(x,t)$. In this case, the standard convention for denoting differentiation with respect to one variable while the other is held constant (which is the case here—we're looking at the sum of forces at one instant of time) is to replace d/dx with $\partial/\partial x$.

So we should write:

$$\vec{F} = T\frac{\partial^2 y}{\partial x^2}\Delta x .$$

The final piece of the puzzle is the acceleration of the bit of string: in our small amplitude approximation, it's only moving up and down, that is, in the *y*-direction—so the acceleration is just $\partial^2 y/\partial t^2$, and canceling Δx between the mass $m = \mu\Delta x$ and $\vec{F} = T\frac{\partial^2 y}{\partial x^2}\Delta x$, $\vec{F} = m\vec{a}$ gives:

$$T\frac{\partial^2 y}{\partial x^2} = \mu\frac{\partial^2 y}{\partial t^2}.$$

This is called the *wave equation*.

It's worth looking at this equation to see why it is equivalent to $\vec{F} = m\vec{a}$. Picture the graph $y = y(x,t)$, showing the position of the string at the instant *t*. At the point *x*, the differential $\partial y/\partial x$ is the *slope* of the string. The *second* differential, $\partial^2 y/\partial x^2$, is the rate of change of the slope—in other words, how much the string is *curved* at *x*. And, it's this curvature that ensures the $\vec{T}$'s at the two ends of a bit of string are pointing along slightly different directions, and therefore don't cancel. This force, then, gives the mass×acceleration on the right.

Solving the Wave Equation

Now that we've derived a wave equation from analyzing the motion of a tiny piece of string, we must check to see that it is consistent with our previous assertions about waves, which were based on experiment and observation. For example, we stated that a wave traveling down a rope kept its shape, so we could write $y(x,t) = f(x - vt)$. Does a general function $f(x - vt)$ necessarily satisfy the wave equation? This *f* is a function of a single variable, let's call it $u = x - vt$. On putting it into the wave equation, we must use the chain rule for differentiation:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial f}{\partial u}, \quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} = -v\frac{\partial f}{\partial u}$$

and the equation becomes

$$T\frac{\partial^2 f}{\partial u^2} = \mu v^2\frac{\partial^2 f}{\partial u^2}$$

so the function $f(x - vt)$ will always satisfy the wave equation provided

$$v^2 = \frac{T}{\mu}.$$

All traveling waves move at the same speed—and the speed is determined by the tension and the mass per unit length. We could have figured out the equation for v^2 dimensionally, *but* there would have been an overall arbitrary constant. We need the wave equation to prove that constant is 1.

Incorporating the above result, the equation is often written:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

Of course, waves can travel both ways on a string: an arbitrary function $g(x+vt)$ is an equally good solution.

The Principle of Superposition

The wave equation has a very important property: if we have two solutions to the equation, then the sum of the two is *also* a solution to the equation. It's easy to check this:

$$\frac{\partial^2 (f+g)}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 g}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} + \frac{1}{v^2}\frac{\partial^2 g}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 (f+g)}{\partial t^2}.$$

Any differential equation for which this property holds is called a linear differential equation: note that $af(x,t)+bg(x,t)$ is also a solution to the equation if *a*, *b* are constants. So you can add together—superpose—multiples of any two solutions of the wave equation to find a new function satisfying the equation.

Harmonic Traveling Waves

Imagine that one end of a long taut string is attached to a simple harmonic oscillator, such as a tuning fork—this will send a harmonic wave down the string,

$$f(x-vt) = A\sin k(x-vt).$$

The standard notation is

$$f(x-vt) = A\sin(kx-\omega t)$$

where of course

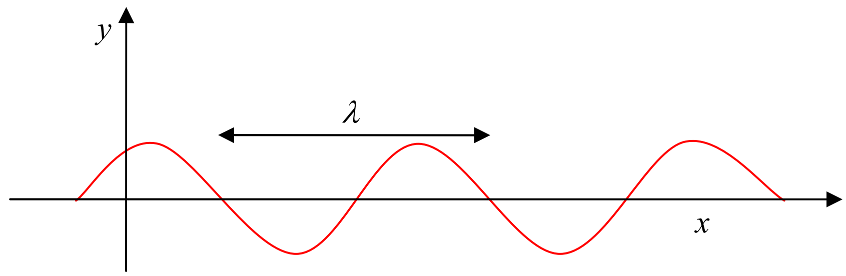
$$\omega = vk.$$

More notation: the wavelength of this traveling wave is λ, and from the form $A\sin(kx-\omega t)$, at say $t=0$,

$$k\lambda = 2\pi .$$

At a fixed *x*, the string goes up and down with frequency given by $\sin\omega t$, so the frequency *f* in cycles per second (Hz) is

$$f = \frac{\omega}{2\pi} \text{ Hz.}$$



Now imagine you're standing at the origin watching the wave go by. You see the string at the origin do a complete up-and-down cycle *f* times per second. Each time it does this, a whole wavelength of the wave travels by. Suppose that at $t = 0$ the wave, coming in from the left, has just reached you.

Then at $t = 1$ second, the front of the wave will have traveled *f* wavelengths past you—so the speed at which the wave is traveling

$$v = \lambda f \text{ meters per second.}$$

Energy and Power in a Traveling Harmonic Wave

If we jiggle one end of a string and send a wave down its length, we are obviously supplying energy to the string—for one thing, as the wave moves down, bits of the string begin moving, so there is kinetic energy. And, there's also potential energy—remember the wave won't go down at all unless there is tension in the string, and when the string is waving it's obviously longer than when it's motionless along the *x*-axis. This stretching of the string takes work against the tension *T* equal to force times distance, in this case equal to the force *T* multiplied by the distance the string has been stretched. (We assume that this increase in length is not sufficient to cause significant increase in *T*. This is usually ok.)

For the important case of a *harmonic* wave traveling along a string, we can work out the energy per unit length exactly. We take

$$y(x,t) = A\sin(kx - \omega t).$$

If the string has mass μ per unit length, a small piece of string of length Δx will have mass $\mu\Delta x$, and moves (vertically) at speed $\partial y/\partial t$, so has kinetic energy $(1/2)\mu\Delta x(\partial y/\partial t)^2$, from which the kinetic energy of a length of string is

$$K.E. = \int \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 dx.$$

For the harmonic wave $y(x,t) = A\sin(kx - \omega t)$,

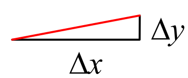
$$K.E. = \int \frac{1}{2}\mu A^2\omega^2 \cos^2(kx - \omega t)dx$$

and since the average value $\overline{\cos^2(kx-\omega t)} = \frac{1}{2}$, for a continuous harmonic wave the average *K.E.* per unit length

$$\overline{K.E.}/\text{meter} = \tfrac{1}{4}\mu\omega^2 A^2.$$

To find the average potential energy in a meter of string as the wave moves through, we need to know how much the string is stretched by the wave, and multiply that length increase by the tension *T*.

Let's start with a small length Δx of string, and suppose that the change in *y* from one end to the other is Δy:



The string (red) is the hypotenuse of this right-angled triangle, so the amount of stretching Δl of this length Δx is how much longer the hypotenuse is than the base Δx.

So

$$\Delta l = \sqrt{(\Delta x)^2 + (\Delta y)^2} - \Delta x = \Delta x\sqrt{1 + (\Delta y/\Delta x)^2} - \Delta x.$$

Remembering that we're only considering small amplitude waves, $\Delta y/\Delta x$ is going to be small, so we can expand the square root using the result

$$\sqrt{1+x} \cong 1 + \tfrac{1}{2}x \quad \text{for small } x$$

to find

$$\Delta l \cong \tfrac{1}{2}(\Delta y/\Delta x)^2 \Delta x.$$

To find the total stretching of a unit length of string, we add all these small stretches, taking the limit of small Δx's to find

$$P.E./\text{meter} = \int \tfrac{1}{2} T\left(\partial y/\partial x\right)^2 dx = \int \tfrac{1}{2} TA^2k^2\cos^2\left(kx-\omega t\right)dx.$$

Now, just as for the kinetic energy discussed above, since $\overline{\cos^2(kx-\omega t)} = \tfrac{1}{2}$, the average potential energy per meter of string is

$$\overline{P.E.}/\,\text{meter} = \tfrac{1}{4}Tk^2A^2 = \tfrac{1}{4}\mu\omega^2A^2, \text{ since } \omega = vk \text{ and } v^2 = T/\mu.$$

That is to say, the average potential energy is the same as the average kinetic energy. This is a very general result: it is true for all harmonic oscillators (excepting the case of heavy damping).

Finally, the ***power*** in a wave traveling down a string is the rate at which it delivers energy at its destination. Adding together the kinetic and potential energy contributions above,

$$\overline{\text{total energy}}/\,\text{meter} = \tfrac{1}{2}\mu\omega^2A^2.$$

Now, if the wave is traveling at v meters per second, and being totally absorbed at its destination (the end of the string) the energy delivered to that end in one second is all the energy in the last v meters of the string. By definition, this is the *power*: the energy delivered in joules per second, That is,

$$\text{power} = \tfrac{1}{2}v\mu\omega^2A^2.$$

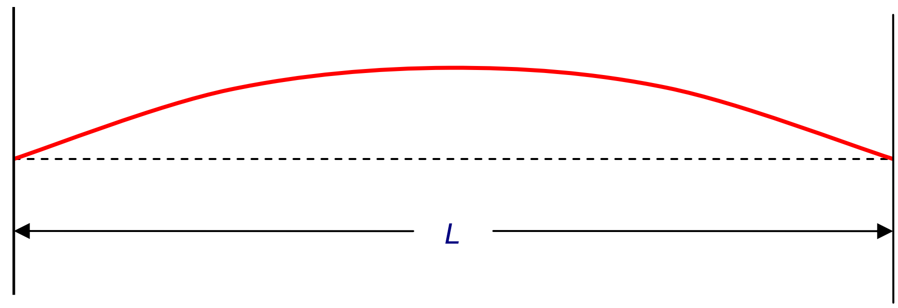
Standing Waves from Traveling Waves

An amusing application of the principle of superposition is adding together harmonic traveling waves moving in opposite directions to get a standing wave:

$$A\sin\left(kx-\omega t\right) + A\sin\left(kx+\omega t\right) = 2A\sin kx\cos\omega t\,.$$

You can easily check that $2A\sin kx\cos\omega t$ *is* a solution to the wave equation (provided $\omega = vk$, of course) and it is always zero at points x satisfying $kx = n\pi$, so for a string of length L, fixed at the two ends, the appropriate k are given by $kL = n\pi$.

The longest wavelength standing wave for a string of length L fixed at both ends has wavelength $\lambda = 2L$, and is termed the *fundamental*.



Fundamental Mode of Vibration of a String Fixed at Both Ends

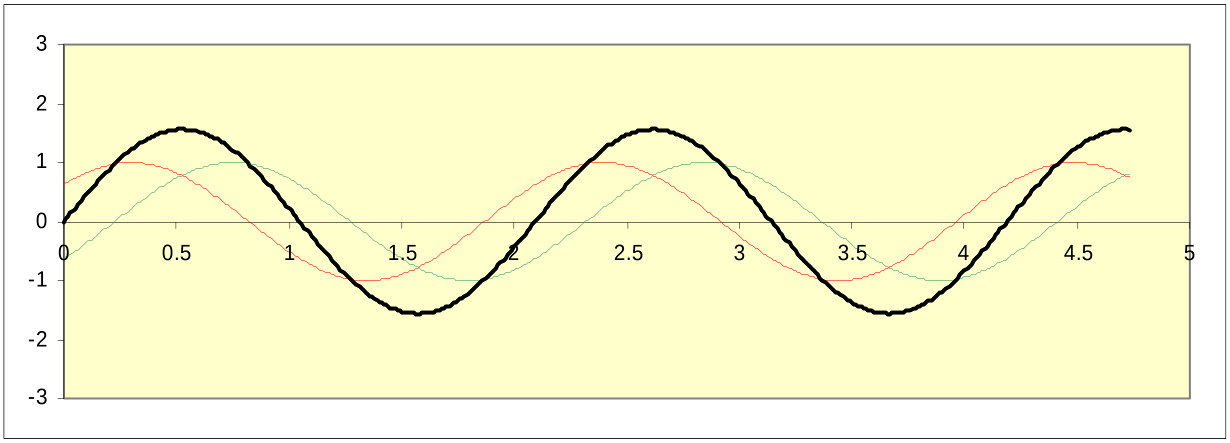
The x-dependence of this wave, $\sin kx$, is clearly $\sin(\pi x/L)$, so $k = \pi/L$.

The radial frequency of the wave is given by $\omega = vk$, so $\omega = v\pi/L$, and the frequency in cycles per second, or Hz, is

$$f = \omega/2\pi = v/2L \text{ Hz.}$$

(This is the same as the frequency $f = v/\lambda$ of a traveling wave having the same wavelength.)

Here's a realization of the superposition of two traveling waves to form a standing wave using a spreadsheet:



Here the red wave is $A\sin(kx-\omega t)$ and moves to the right, the green $A\sin(kx+\omega t)$ moves to the left, the black is the sum of the two and its oscillations stay in place.

But this represents just one instant! To see the full development in time—which you need to do to get real insight into what's going on—download the spreadsheet from http://galileo.phys.virginia.edu/classes/152.mf1i.spring02/WaveSum.xls ,then click and hold at the end of the slider bar to animate.

Exercise: What do you think the black wave will look like if the red and green have different amplitudes? Try it on the spreadsheet.

Boundary Conditions: at the End of the String

Adding Opposite Pulses

Our first move in working with waves was to jiggle the end of a string (or spring) and generate a pulse that we saw traveled along with no perceptible change in shape. We showed that our observation could be expressed mathematically: taking the string initially at rest along the x-axis, its displacement y at point x at time t was evidently described by a function of the form $y = f(x - vt)$. This function keeps its shape, but as t progresses it moves to the right with speed v.

We next analyzed the dynamics of the vibrating string by applying Newton's Laws of Motion to a little bit of string. This reveals an equation, the **wave equation**, that *any* vibration of the string must obey. Reassuringly, our observed form for the moving pulse, $y = f(x - vt)$, does in fact satisfy the wave equation.

The wave equation has one very important property: *if you add two solutions to the wave equation, the sum is another solution to the wave equation*. This means that if you and a friend send pulses down a rope from the opposite end, the pulses will go right through each other, and when they're on top of each other, the total displacement of the rope will be just the sum of the displacements corresponding to the individual pulses. We shall see that this gives an important clue for understanding what happens when a pulse reaches the end of the string.

Pulse Reflection

What happens when the pulse gets to the end of the string depends on the end of the string: there are two possibilities:

(a) the end of the string is fixed,
(b) the end of the string is free to move up and down (the pulse corresponds to the string moving in an up-down way).

We refer to these as *fixed end* and *free end* boundary conditions. You may be wondering how the string could have a free end, since it needs to be under tension for the wave to propagate at all. This is arranged by having the string terminate on a ring which is free to move up and down a smooth rod perpendicular to the direction of the string. More important examples of free-end vibrations come up in analyzing musical instruments like the organ, where we shall find that a closed end to an organ pipe is equivalent to a fixed end, an open pipe end is a free end.

An Experiment on Fixed End Reflection and Free End Reflection

We use a demonstration which has a wire under tension with thin parallel rods perpendicular to the wire attached to it at their centers. The ends of these rods are painted white for visibility, and waves will travel down this array sufficiently slowly to be followed easily. It's easy to send a pulse down from one end, then either hold the other end fixed or let it move freely, and observe what happens when the pulse reaches the end.

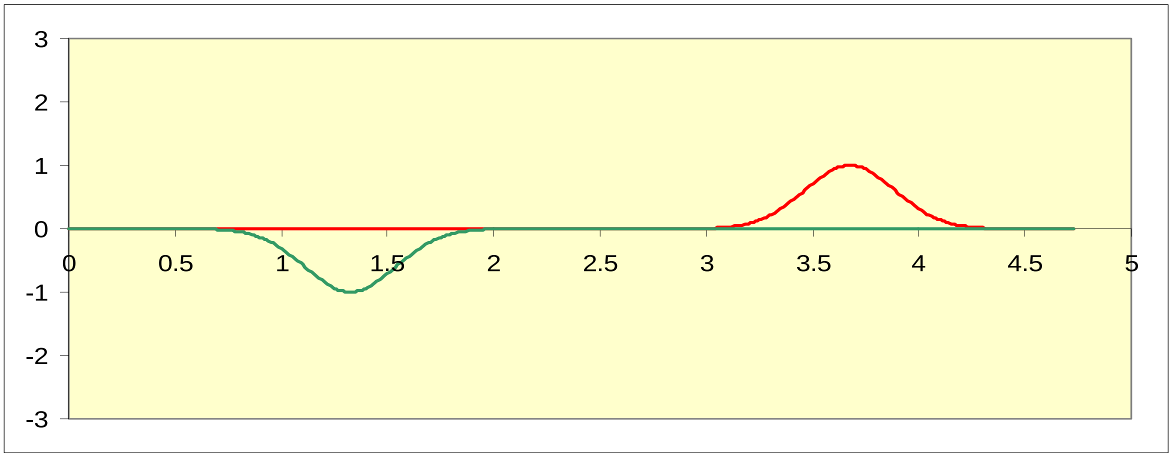
It is found that when the pulse reaches the fixed end, it is reflected with its shape intact, *but switched in sign*: if before reflection the pulse bulged the string in the +*y* direction, after reflection it bulged the string in the –*y* direction.

However, if the end rod is free to rotate, the pulse is reflected *without* a change of sign.

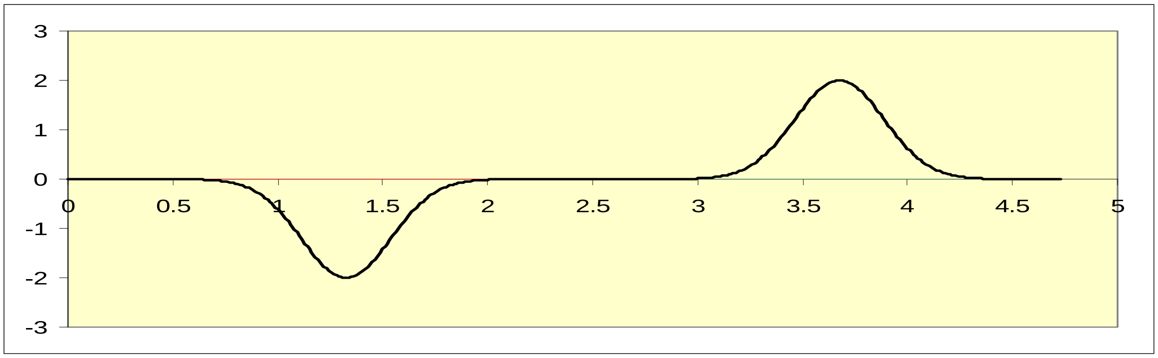
Understanding Sign Change in Pulse Reflection

The key to seeing what's going on when a pulse is reflected is to do a different experiment: send two pulsed down a rope from opposite ends and watch carefully as they pass in the middle. Let's start with two pulses identical in shape, but of opposite sign. We'll generate the pulses with a spreadsheet, and watch them as they pass. Remember, the total displacement of the string at any point is the sum of the displacements of the separate pulses.

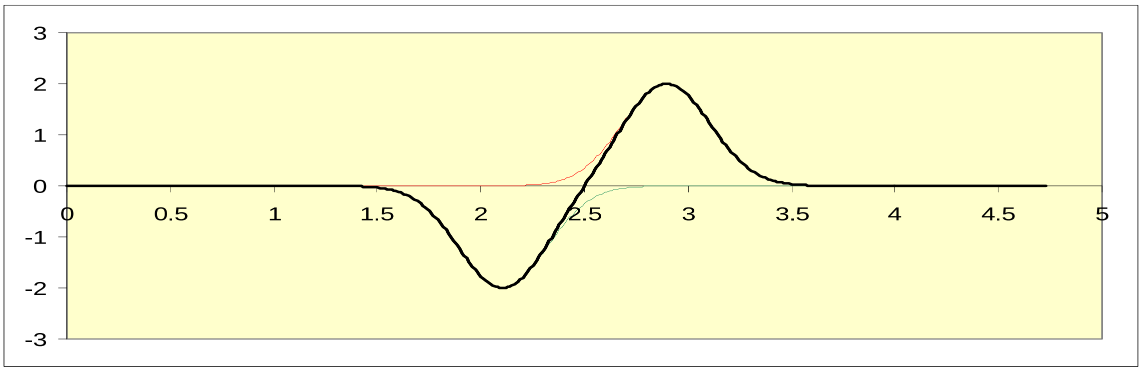
The two separate pulses look like this:



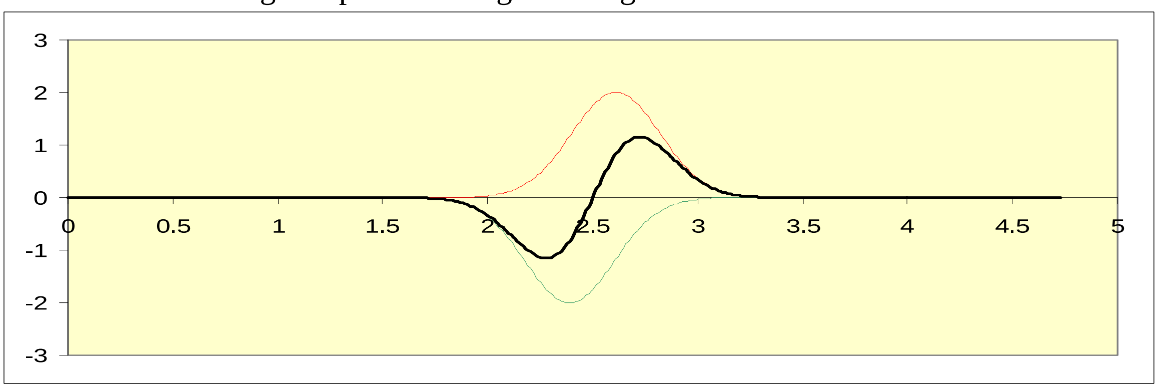
Bearing in mind that the green hides the red along the axis when they're together, the green and red are separately solutions to the wave equation, the sum of the two pulses is also a solution, and that's the black line—the actual position of the string at some moment after the pulses are sent on their way—in the diagram below:



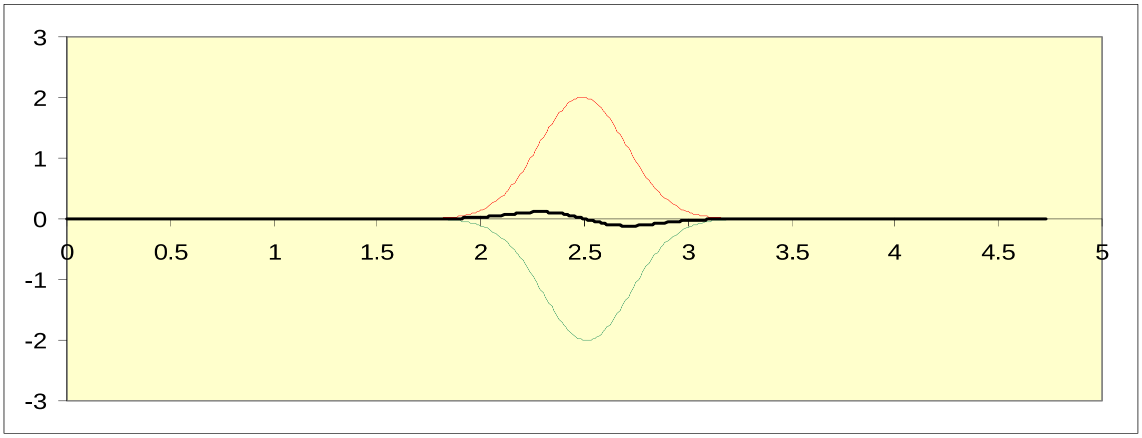
Tracking the two pulses as time goes on, they meet:

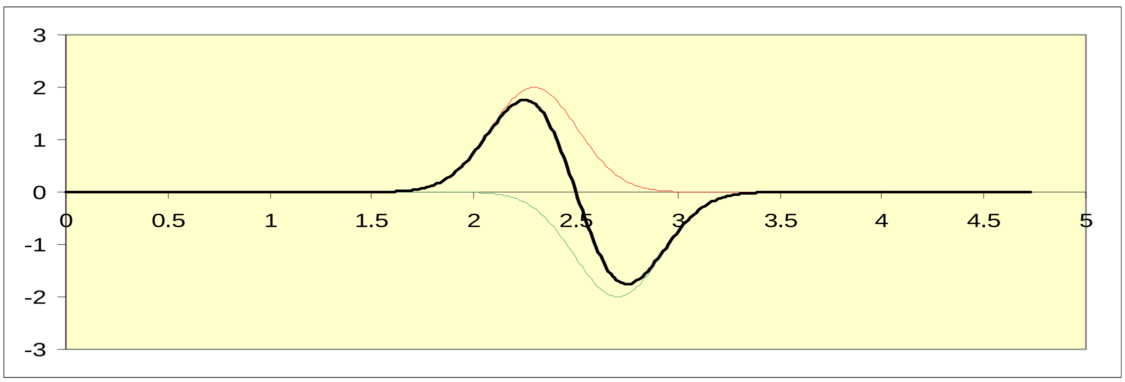


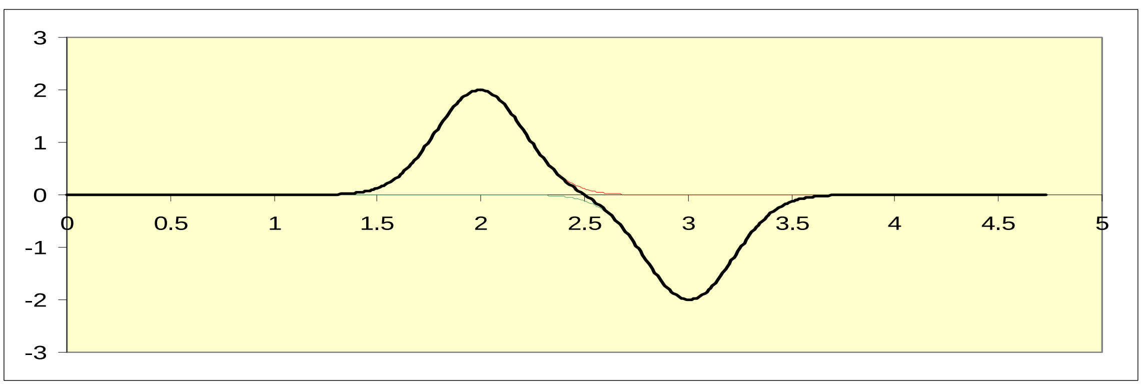
Now we can see the green pulse moving to the right and the red to the left:



They pass (look at the string!):







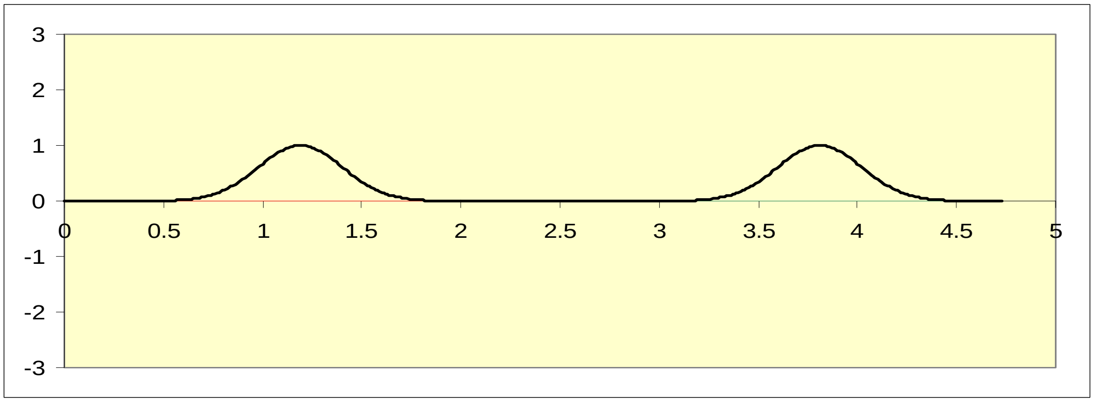
Reviewing this sequence of pictures, notice that the string at the central point, $x = 2.5$, *never moves*. It might as well have been nailed in place. Imagine we *did* nail it in place, chopped off the string to the right of center, and just sent one pulse down from the left-hand end. To see what would happen, we would need to solve the wave equation for the string subject to the fixed end on the right, and a pulse being sent down from the left. But we already know the solution to the wave equation for the whole string with the initial two pulses described above, and the midpoint always stayed fixed. This solution, confined to the left hand half, is to the same equation with the same boundary condition and the same initial configuration as the two pulses on the whole string scenario—so it must be the same solution! We are forced to conclude that when a pulse that curves *downwards* is sent towards a fixed end, the reflected pulse curves *upwards*—it just follows the same sequence as the left-hand half in our two pulses full string solution. And, needless to add, this is what we see experimentally.

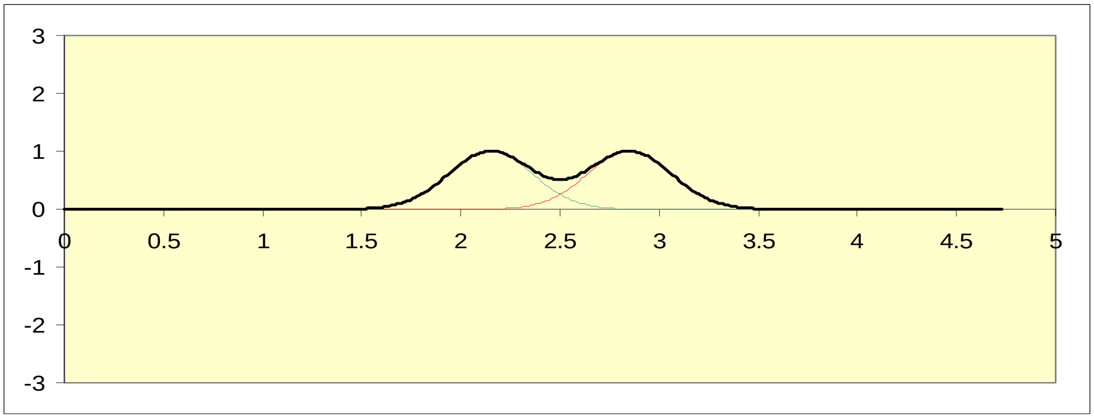
Free End Boundary Condition

Suppose now in the rod model we send a pulse down from the left, but instead of fixing the rod on the right-hand end, we allow it to rotate freely. What happens? Recall that in deriving the wave equation by writing $F = ma$ for a small piece of string, the accelerating force on the string depended on the small difference in slope of the string at the two ends of the little piece under consideration. Our rod model is a discretized version: for a rod somewhere in the middle, the net force on it depends on the slight difference in slope of the lines connecting it to its neighbors. *But* for the last rod, if it's free to move, there's only a force on one side. For it to have the same acceleration as its neighbors, the force it feels from its only neighbor must be tiny, it must be comparable to the *difference* in force from typical neighbor rods. This means that the curve made by the dots on the ends of the rods (see photo of equipment) must be essentially horizontal at the end—the last two rods are almost lined up.

A string is the limit of this picture with more and more rods, closer and closer together. The *free end boundary condition for a string is, then, that its slope goes to zero at the boundary*.

It's easy to see that with this boundary condition, a pulse will be reflected *without change of sign*. Just take the spreadsheet and send down two pulses of the same sign:





It's evident from the complete symmetry that the slope of this curve is always going to be zero at the central point, so if we blind ourselves to the right-hand half, and imagine the left-hand half to be a complete string subject to the boundary condition at the right-hand end that the string slope be zero, a pulse coming towards the end from the left will be reflected *without change of sign*.

As mentioned earlier, although this *is* a rather artificial boundary condition for a string, we shall soon see it is exactly the right boundary condition for an open end of an organ pipe, so this analysis is relevant for some real-life systems.

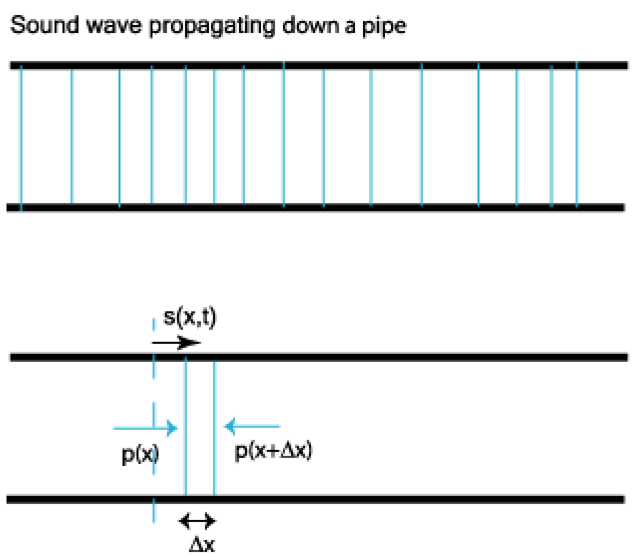
Sound Waves

"One-Dimensional" Sound Waves

We'll begin by considering sound traveling down a hollow pipe, to avoid unnecessary mathematical complications. Sound is a longitudinal wave—as the wave passes through, the air moves backwards and forwards in the pipe, this oscillatory movement is in the same direction the wave is traveling.

To visualize what's happening, imagine mentally dividing the air in the pipe, which is at rest if there is no sound, into a stack of thin slices. Think about one of these slices. In equilibrium, it feels equal and opposite pressure from the gas on its two sides. (This is analogous to the little bit of string at rest feeling equal and opposite tension on its two sides, but of course the gas pressure is inward). As the sound wave goes through, the pressure wave generates slight differences in pressure on the two sides of our thin slice of air, and this imbalance of forces causes the slice to accelerate.

To analyze this quantitatively—to apply $\vec{F} = m\vec{a}$ to the thin slice of air—we must begin by defining *displacement*, the quantity corresponding to the string's transverse movement $y(x,t)$. We shall use $s(x,t)$ to denote the *horizontal* (along the pipe) displacement of the thin slice of air which rests at position x when no sound is present.



External forces acting to accelerate a "slice of air" displaced from its equilibrium position by s(x,t). (The displacement shown is greatly exaggerated.)

If the pipe has radius a, and hence cross-sectional area πa^2, a slice of air of thickness Δx has volume $\pi a^2 \Delta x$, so writing the density of air ρ (1.29 kg/m^3), the mass of the slice of air is $m = \rho V = \rho \pi a^2 \Delta x$. Clearly, its acceleration is $a = \partial^2 s(x,t)/\partial t^2$, so we already have the right-hand side of $\vec{F} = m\vec{a}$. To find the left hand side—the force on the thin slice of air—we must find the pressure imbalance between the two sides.

Relating Pressure Change to How the Displacement Varies

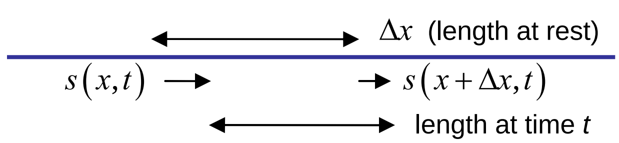
The pressure change as the sound wave moves down the tube is directly tied to the local compression or expansion of the gas. It's like a spring: as the gas is compressed into a smaller volume, its pressure rises, and as the gas expands the pressure drops. And, exactly as for a spring, the changes in pressure and volume are linearly related. The coefficient of proportionality is called the ***bulk modulus***, usually written B, and defined by the equation:

$$\Delta p = -B\frac{\Delta V}{V}$$

Note the sign! As the volume decreases, the pressure increases. Since the ratio of volumes is dimensionless, the units for the bulk modulus are the same as for pressure: Pascals. For air at standard temperature and pressure, the bulk modulus $B = 10^5$ Pa.

Now, we are tracking the motion of the gas as the sound wave passes through by following the parameter $s(x,t)$, the displacement along the tube at time t of gas having equilibrium position x. Obviously, if $s(x,t)$ does not depend on x, all the gas is shifted by the same amount, and no compression or expansion has taken place. Local change in volume *only* happens if there is local *variation* in $s(x,t)$.

To make this quantitative, consider a slice of gas having thickness Δx (when at rest): if, at some instant when the sound wave is passing through, the right-hand end is displaced by $s(x+\Delta x,t)$, and the left-hand end by a greater amount $s(x,t)$, say,



Compression of a thin "slice" of air resulting from different displacements at the two ends

the thickness of the slice has evidently been changed from Δx to

$$\Delta x - \left(s(x,t) - s(x+\Delta x,t)\right).$$

Since the volume of air in the slice is directly proportional to its thickness, the sound wave has at this instant changed the *volume* of the air initially in the segment Δx near the point x by a fraction

$$\frac{\Delta V}{V}=\frac{s\left(x+\Delta x,t\right)-s\left(x,t\right)}{\Delta x}=\frac{\partial s\left(x,t\right)}{\partial x}$$

the differential being exact in the limit of a thin slice.

Therefore, *the local extra pressure is directly proportional to minus the gradient of* $s\left(x,t\right)$:

$$\Delta p=-B\frac{\Delta V}{V}=-B\frac{\partial s(x,t)}{\partial x}.$$

From *F* = *ma* to the Wave Equation

Having found how the local pressure variation relates to $s\left(x,t\right)$, we're ready to derive the wave equation from $F = ma$ for a thin slice of gas. Recall that for such a slice $m=\rho V=\rho\pi a^2\Delta x$, and of course $a=\partial^2 s\left(x,t\right)/\partial t^2$.

The net force F on the slice is the difference between the pressure at x and that at $x+\Delta x$:

$$F=p(x,t)\pi a^2-p(x+\Delta x,t)\pi a^2=-\pi a^2 B\frac{\partial s(x,t)}{\partial x}+\pi a^2 B\frac{\partial s(x+\Delta x,t)}{\partial x}=\pi a^2 B\Delta x\frac{\partial^2 s(x,t)}{\partial x^2}.$$

Putting this into $F = ma$:

$$\frac{\partial^2 s(x,t)}{\partial x^2}=\frac{1}{v^2}\frac{\partial^2 s(x,t)}{\partial t^2},\text{ where } v=\sqrt{\frac{B}{\rho}}$$

This is exactly the wave equation we found for the string, with now the longitudinal displacement s replacing the transverse displacement y, and the bulk modulus playing the role of the string tension, both being measures of stored potential energy arising from local variations in displacement. The densities, of course, play the same role in the two cases, measuring how much *kinetic* energy is stored for given local displacement velocities.

Boundary Conditions for Sound Waves in Pipes

Since the new wave equation is identical in form to that for waves on a string, our discussion of traveling waves, standing waves, etc., for a string can be carried over with the appropriate changes of notation and applied here.

For example, a standing wave in a pipe has the form $s\left(x,t\right)=A\sin kx\sin\omega t$, this would be for a pipe *closed* at $x = 0$, so that the air doesn't move at $x = 0$.

The boundary condition for a closed end of a pipe is:

$$s(x,t) = 0 \text{ at a closed end.}$$

What about an *open* end? In that case, the air is free to move—the boundary condition won't be $s(x,t) = 0$. However, the pressure is *not* free to vary: it's atmospheric pressure, the pipe being open to the atmosphere. So at an open end $\Delta p = 0$. Remembering that $\Delta p = -B\partial s(x,t)/\partial x$, the boundary condition is:

$$\frac{\partial s(x,t)}{\partial x} = 0 \text{ at an open end.}$$

Harmonic Standing Waves in Pipes

Consider now a standing harmonic wave in a pipe of length L, *closed* at $x = 0$ but *open* at $x = L$.

From the $x = 0$ boundary condition, the wave must have the form $s(x,t) = A\sin kx \sin \omega t$.

The $x = L$ open end boundary condition requires that the slope $\partial s(L,t)/\partial x = 0$.
That is, $\cos kL = 0$.

Exercise: Prove that the longest wavelength standing wave possible in the pipe has wavelength $4L$, and sketch the wave.

Exercise: what is the *next* longest wavelength of a possible standing wave in the pipe? Draw a picture.

Traveling Waves: Power and Intensity

Another solution to the wave equation is

$$s(x,t) = A\sin(kx - \omega t)$$

where $\omega = vk$, just as for string. This is a wave traveling down the pipe. It could be generated by an oscillating plate at the closed end: in other words, a speaker.

How much **power** is this speaker putting out? It's moving and pushing against the pressure:

Power = *P* = rate of working = force x velocity = pressure x area x velocity

How fast is it moving? At time t, the plate is at

$$s(x = 0, t) = -A\sin \omega t,$$

so it is moving at velocity

$$v_{plate}\left(t\right)=\frac{\partial s\left(x=0,t\right)}{\partial t}=-A\omega\cos\omega t.$$

The pressure at the plate is Δp where

$$\Delta p=-B\frac{\partial s(x,t)}{\partial x}=-B\frac{\partial}{\partial x}A\sin\left(kx-\omega t\right)=-ABk\cos\omega t$$

at $x = 0$.

So the rate of working at time t, the power $P(t)$ = velocity x force:

$$P\left(t\right)=v_{plate}\left(t\right)\Delta p\pi a^{2}=A^{2}B\pi a^{2}\omega k\cos^{2}\omega t$$

The standard definition of power for any kind of wave generator is the ***average*** *power over a complete cycle.*

Since the average value of $\cos^2 x$ = ½,

$$\text{power}\ \ P=\tfrac{1}{2}A^{2}B\pi a^{2}\omega k.$$

Using $B=v^{2}\rho$ and $\omega=vk$, this can be written

$$P=\tfrac{1}{2}A^{2}\pi a^{2}\omega^{2}\rho v.$$

This also tells us how much energy there is in the wave as it travels:

$$\tfrac{1}{2}A^{2}\pi a^{2}\omega^{2}\rho\ \text{ per meter.}$$

The ***intensity*** of the wave is ***average power per square meter of cross sectional area***, so here

$$\text{Intensity}\ \ I=\tfrac{1}{2}A^{2}\omega^{2}\rho v$$

and I is measures in *watts per square meter*.

The factor v, the velocity, in the above expression comes about because in one second, the energy delivered by a steady sound wave to one square meter of area perpendicular to the direction of the wave's motion is the energy in v cubic meters of wave: taking the speed of sound to be 330 meters per second, 330 cubic meters of sound energy will plough into one square meter each second.

Waves in Two and Three Dimensions

Introduction

So far, we've looked at waves in one dimension, traveling along a string or sound waves going down a narrow tube. But waves in higher dimensions than one are very familiar—water waves on the surface of a pond, or sound waves moving out from a source in three dimensions.

It is pleasant to find that these waves in higher dimensions satisfy wave equations which are a very natural extension of the one we found for a string, and—very important—they also satisfy the *Principle of Superposition*, in other words, if waves meet, you just add the contribution from each wave. In the next two paragraphs, we go into more detail, but this Principle of Superposition is the crucial lesson.

The Wave Equation and Superposition in One Dimension

For waves on a string, we found Newton's laws applied to one bit of string gave a differential wave equation,

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

and it turned out that *sound waves in a tube satisfied the same equation*. Before going to higher dimensions, I just want to focus on one crucial feature of this wave equation: it's *linear*, which just means that if you find two different solutions $y_1(x,t)$ and $y_2(x,t)$ then $y_1(x,t)+y_2(x,t)$ is also a solution, as we proved earlier.

This important property is easy to interpret *visually*: if you can draw two wave solutions, then at each point on the string simply add the displacement $y_1(x,t)$ of one wave to the other $y_2(x,t)$—you just add the waves together—this also is a solution. So, for example, as two traveling waves moving along the string in opposite directions meet each other, the displacement of the string at any point at any instant is just the sum of the displacements it would have had from the two waves singly. This simple addition of the displacements is termed "interference", doubtless because if the waves meeting have displacement in opposite directions, the string will be displaced less than by a single wave. It's also called the ***Principle of Superposition***.

The Wave Equation and Superposition in More Dimensions

What happens in higher dimensions? Let's consider two dimensions, for example waves in an elastic sheet like a drumhead. If the rest position for the elastic sheet is the (x, y) plane, so when it's vibrating it's moving up and down in the z-direction, its configuration at any instant of time is a function $z(x,y,t)$.

In fact, we could do the same thing we did for the string, looking at the total forces on a little bit and applying Newton's Second Law. In this case that would mean taking one little bit of the drumhead, and instead of a small stretch of string with tension pulling the two ends, we would have a small *square* of the elastic sheet, with tension pulling all around the edge. Remember that the net force on the bit of string came about because the string was curving around, so the tensions at the opposite ends tugged in slightly different directions, and didn't cancel. The $\partial^2 y/\partial x^2$ term measured that curvature, the rate of change of slope. In two dimensions, thinking of a small square of the elastic sheet, things are more complicated. Visualize the bit of sheet to be momentarily like a tiny patch on a balloon, you'll see it curves in two directions, and tension forces must be tugging all around the edges. The total force on the little square comes about because the tension forces on opposite sides are out of line if the surface is curving around, now we have to add *two* sets of almost-opposite forces from the two pairs of sides. I'm not going to go through all the math here, but I hope it's at least plausible that the equation is:

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{1}{v^2}\frac{\partial^2 z}{\partial t^2}.$$

The physics of this equation is that the acceleration of a tiny bit of the sheet comes from out-of-balance tensions caused by the sheet curving around in *both* the *x*- and *y*-directions, this is why there are the two terms on the left hand side.

Remarkably, this *same equation* comes out for water waves (at least for small amplitudes), sound waves, and even the electromagnetic waves we now know as light, radio, etc.. (And, going to three dimensions is easy: add one more term to get $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=\frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$. This sum of partial differentiations is so common in physics that there's a shorthand: $\nabla^2 f=\left(1/v^2\right)\partial^2 f/\partial t^2$.)

Just as we found in one dimension traveling harmonic waves $f\left(x-vt\right)=A\sin\left(kx-\omega t\right)$, with $\omega=vk$, you can verify that the *three*-dimensional equation has harmonic solutions $f\left(x,y,z,t\right)=A\sin\left(k_x x+k_y y+k_z z-\omega t\right)$ and now $\omega=v\left|k\right|$, where $\left|k\right|=\sqrt{k_x^2+k_y^2+k_z^2}$. In fact, $\vec{k}$ is a vector in the direction the wave is moving. The electric and magnetic fields in a radio wave or light wave have just this form (or, closer to the source, a very similar equivalent expression for outgoing spheres of waves, rather than plane waves).

It's important to realize that this more complicated equation is still a *linear* equation—***the principle of superposition still holds***. If two waves on an elastic sheet, or the surface of a pond, meet each other, the result at any point is given by simply adding the displacements from the individual waves. (Assuming as always small waves, so the water waves don't fall apart into foam.)

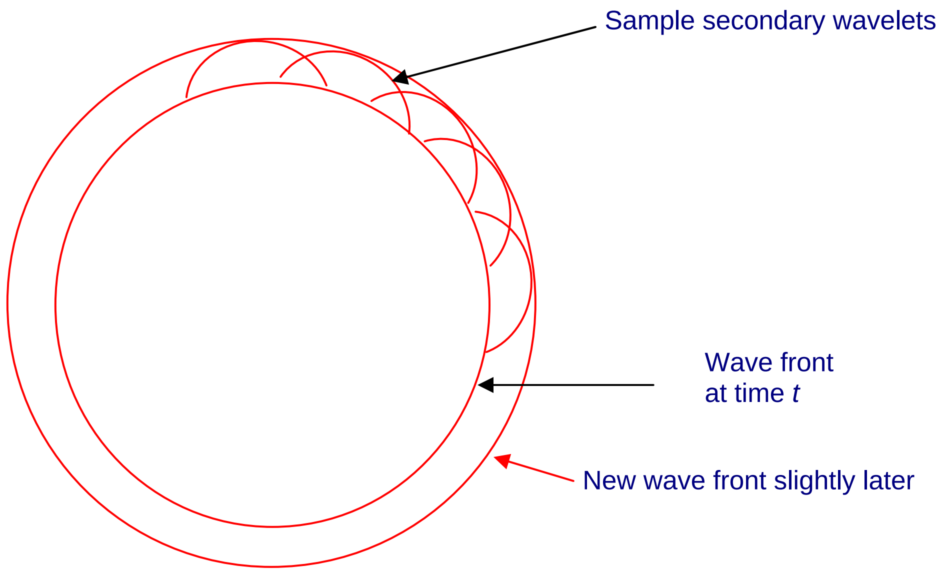
We'll begin by thinking about waves propagating freely in two and three dimensions, than later consider waves in restricted areas, such as a drum head.

How Does a Wave Propagate in Two and Three Dimensions?

A one-dimensional wave doesn't have a choice: it just moves along the line (well, it could get partly reflected by some change in the line and part of it go backwards). But when we go to higher dimensions, how a wave disturbance starting in some localized region spreads out is far from obvious. But we can begin by recalling some simple cases: dropping a pebble into still water causes an outward moving circle of ripples. If we grant that light is a wave, we notice a beam of light changes direction on going from air into glass. Of course, it's not immediately evident that light *is* a wave: we'll talk a lot more about that later.

Huygen's Picture of Wave Propagation

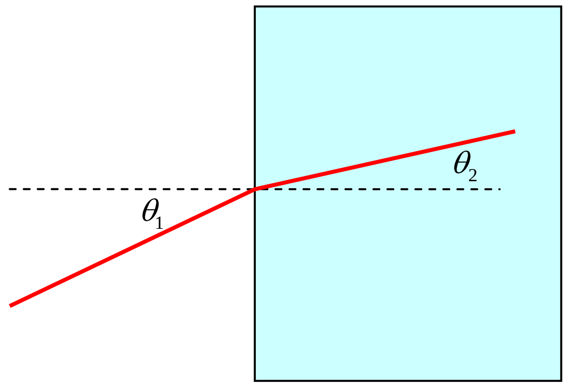
If a point source of light is switched on, the wavefront is an expanding sphere centered at the source. Huygens suggested that this could be understood if at any instant in time each point on the wavefront was regarded as a source of secondary wavelets, and the new wavefront a moment later was to be regarded as built up from the sum of these wavelets. For a light shining continuously, this process just keeps repeating.



Huygens' picture of how a spherical wave propagates: each point on the wave front is a source of secondary wavelets that generate the new wave front.

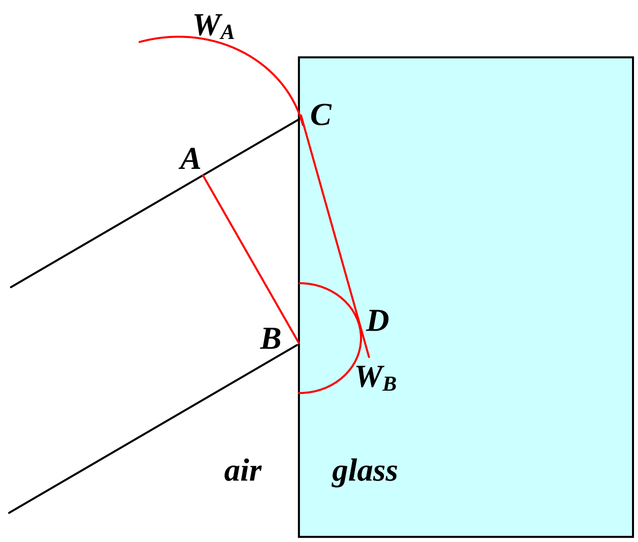
What use is this idea? For one thing, it explains refraction—the change in direction of a wavefront on entering a different medium, such as a ray of light going from air into glass.

If the light moves more slowly in the glass, velocity v instead of c, with $v < c$, then Huygen's picture explains Snell's Law, that the ratio of the sines of the angles to the normal of incident and transmitted beams is constant, and in fact is the ratio c/v.



Snell's Law : a ray of light entering glass from air is bent towards the normal, and $\sin\theta_1 / \sin\theta_2$ is the same for any entering angle.

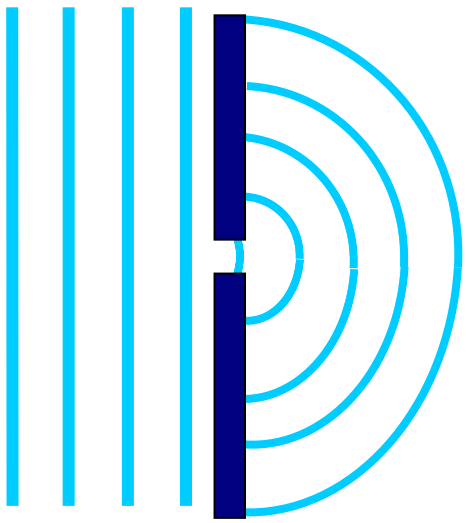
This is evident from the diagram below: in the time the wavelet centered at ***A*** has propagated to ***C***, that from ***B*** has reached ***D***, the ratio of lengths ***AC***/***BD*** being c/v. But the angles in Snell's Law are in fact the angles ***ABC***, ***BCD***, and those right-angled triangles have a common hypotenuse ***BC***, from which the Law follows.



Huygens' explanation of refraction: showing two wavelets from the wavefront ***AB***:

W_B is slowed down compared with ***W_A*** , since it is propagating in glass. This turns the wave front through an angle.

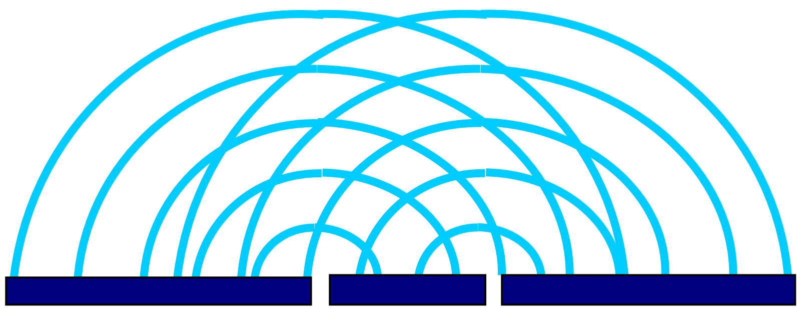
Huygens' picture also provides a ready explanation of what happens when a plane wave front encounters a barrier with one narrow opening: and by narrow, we mean small compared with the wavelength of the wave. It's easy to arrange this for water waves: it's found that on the other side of the barrier, the waves spread out in circular fashion form the small hole.



A plane wave encounters a barrier with an opening smaller than a wavelength: the wave spreads in circular fashion on the far side.

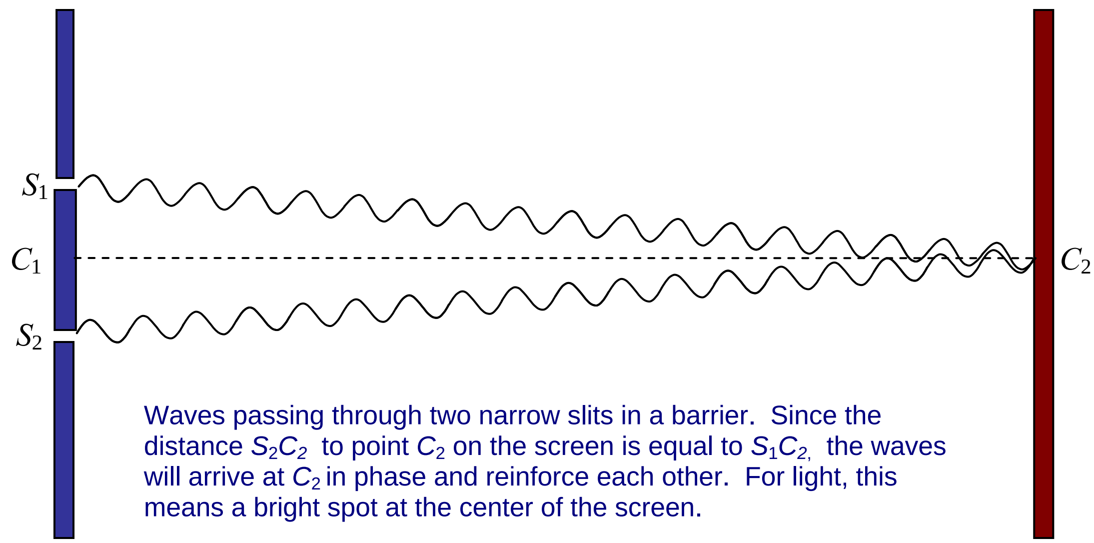
Two-Slit Interference: How Young measured the Wavelength of Light

If the slit is wider than a wavelength or so, the pattern gets more complicated, as we would expect from Huygens' ideas, because now the waves on the far side arise from a line of sources, not what amounts to one point. To investigate this further, consider the simplest possible next case: a barrier with *two* small holes in it, so on the far side we're looking at waves radiating outwards from, effectively, two point sources.

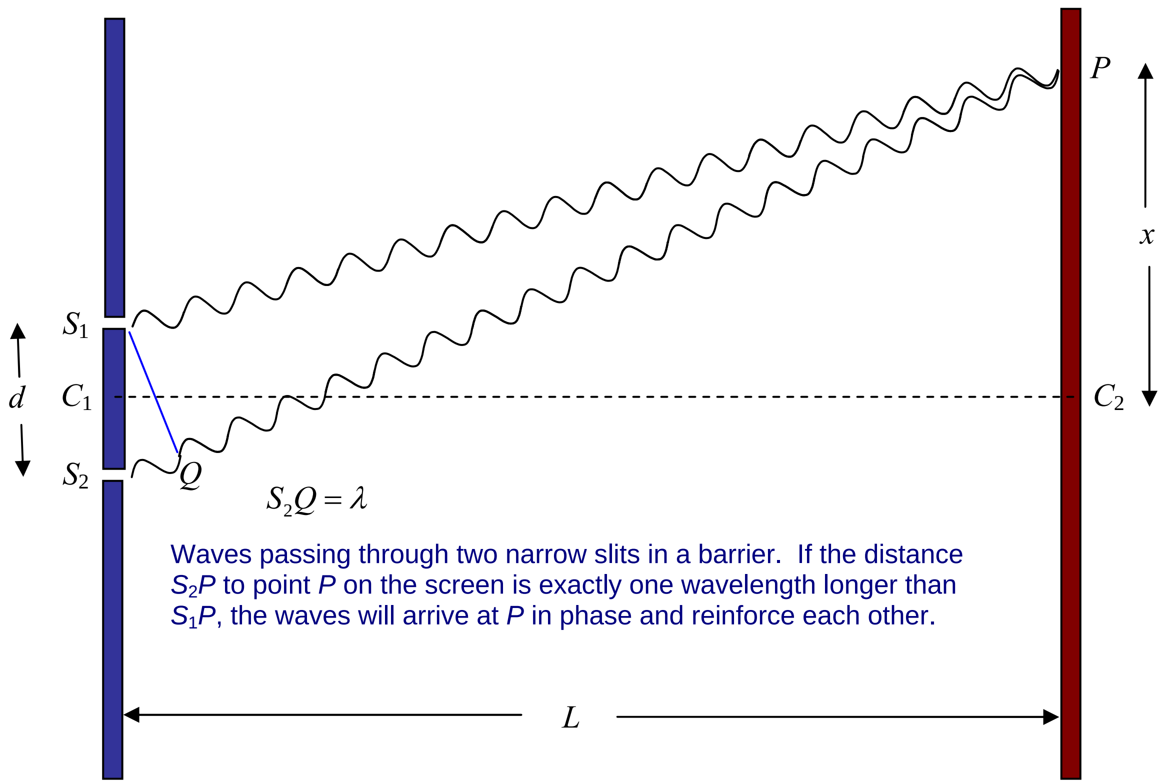


Waves spreading out from two small slits in a barrier: the blue circles represent wave crests, where two cross the wave has maximum positive value, for example along the central line.

For two synchronized sources generating harmonic waves, at any point in the tank equally distant from the two sources (the central line in the picture above), the waves will add, the water will be maximally disturbed. For light waves, there will be a maximum in brightness at the center of a screen as shown in the diagram:



For light waves passing through two narrow slits and shining on a screen (on the right) there will be another bright spot at a point P away from the center C_2 of the screen, *provided the distances of P from the two slits differ by a whole number of wavelengths*:



On the other hand, at a point approximately half way from the center of the screen to P the waves from the two sources will arrive at the screen exactly *out* of phase: the crest of one will arrive with the trough of the other, they will cancel, and there will be no light. Evidently, then, we will see on the screen *a series of bright areas and dark areas*, the brightest spots being at the points where the waves from the two slits arrive exactly in phase.

There is a Flash animation of this pattern formation here.

This pattern, generated by what is called ***interference*** between the waves, and also referred to as a ***diffraction pattern*** is historically important, because it was used to establish that light is a wave, by Thomas Young in 1807. (Recall Newton had believed light was a stream of particles, and that was very widely accepted at the time.)

Young used the pattern to *find the wavelengths* of red and violet light. His method can be understood from the diagram above. We did the experiment in class with a slit separation of about 0.2 mm., giving bright spots on the screen about 3 cm apart, with a screen 10 m from the slits.

That is to say, in the diagram above we had $S_1S_2 = 0.2\times10^{-3}\,\text{m}$, $C_1C_2 = 9.5\,\text{m}$, and we found $C_2P = x = 3$ cm. (within a percent or two). Looking at the diagram, it's clear that the angle to P from the slits is very small, in fact it's $x/L = 3.15\times10^{-3}$ radians. So the diagram as drawn is very exaggerated!

Now, the line S_1Q is perpendicular to the light rays setting off for P (they are *extremely* close to parallel). The angle between S_1Q and S_1S_2 is the same as that between C_1P and C_1C_2, that is, 3.15×10^{-3} radians. This means that the lengths S_1Q and S_1S_2 are effectively equal, and therefore that

$$\frac{S_2Q}{S_1S_2} = \frac{\lambda}{d} = \frac{x}{L} = 3.15\times10^{-3}.$$

This is very accurate for such a small angle, and for the data as given here the wavelength of the light $\lambda = 3.15\times10^{-3}d = 6.3\times10^{-7}\,\text{m} = 630\text{nm}$.

Another Bright Spot

About ten years after Young's result a French civil engineer, Augustin Fresnel, independently developed a wave theory of light, and gave a more complete mathematical analysis. This was disputed by the famous French mathematician Simeon Poisson, who pointed out that if the wave theory were true, one could prove mathematically that in the sharp shadow of a small round object, there would be a bright spot in the center, because the waves coming around the circumference all around would add there. This seemed ridiculous—but French physicist Francois Arago actually did the experiment, and found the spot! The wave theory of light had arrived.

The Doppler Effect

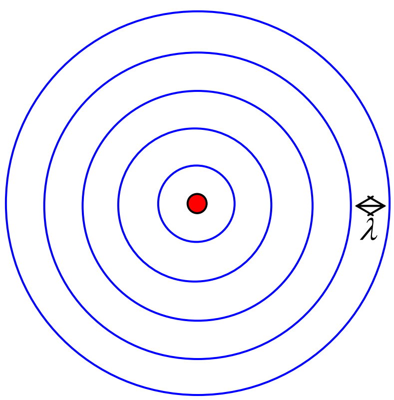
Introduction

(**Flashlet here**)

The Doppler effect is the perceived change in frequency of sound emitted by a source moving relative to the observer: as a plane flies overhead, the note of the engine becomes noticeably lower, as does the siren noise from a fast-moving emergency vehicle as it passes. The effect was first noted by Christian Doppler in 1842. The effect is widely used to measure velocities, usually by reflection of a transmitted wave from the moving object, ultrasound for blood in arteries, radar for speeding cars and thunderstorms. The velocities of distant galaxies are measured using the Doppler effect (the red shift).

Sound Waves from a Source at Rest

To set up notation, a source at rest emitting a steady note generates circular wavecrests:



The concentric circles represent wave crests generated by the central source at a frequency f_0 waves per second.
Their separation is the wavelength λ, where $f_0 = v/\lambda$, v being the speed of the waves.

A stationary observer will (of course) observe them to reach him with frequency f_0.

The circles are separated by one wavelength λ and they travel outwards at the speed of sound v. If the source has frequency f_0, the time interval τ_0 between wave crests leaving the source

$$\tau_0 = \frac{1}{f_0}.$$

As a fresh wave crest is emitted, the previous crest has traveled a distance λ, so, since it's moving at speed v,

$$v\tau_0 = \lambda,$$

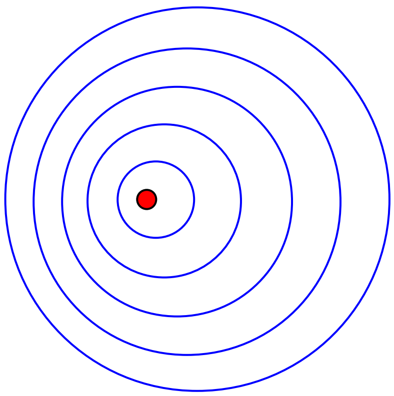
and therefore

$$\lambda f_0 = v.$$

Sound Waves from a Moving Source

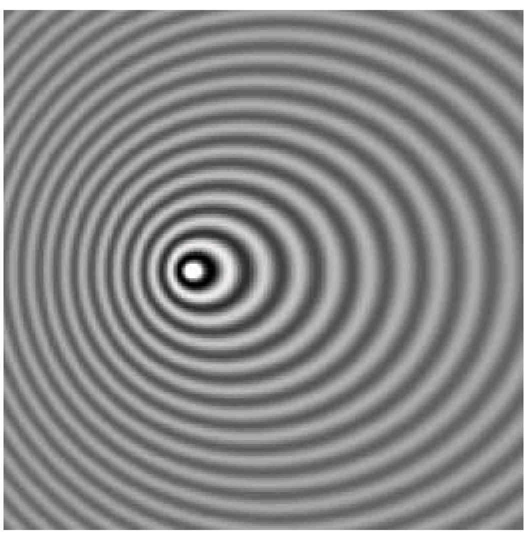
The Doppler effect arises because once a *moving* source emits a circular wave (and provided the source is moving at less than the speed of the wave) the circular wave crest emitted continues its outward expansion *centered on where the source was when it was emitted*, independent of any subsequent motion of the source.

Therefore, if the source is moving at a steady speed, *the centers of the emitted circles of waves will be equally spaced along its path*, indicating its recent history. In particular, if the source is moving steadily to the left, the wave crests will form a pattern:



Wave crests emitted by a source in steady motion to the left at speed u_s.

Or, to be more realistic (from Wikipedia Commons):



It is evident that, as a result of the motion of the source, waves traveling to the left have a shorter wavelength than they had when the source was at rest. And it's easy to understand why.

Denoting the steady source velocity by u_s, in the time $\tau_0 = 1/f_0$ between crests being emitted the source will have moved to the left a distance $u_s\tau_0$. At the same time, the previously emitted crest will itself have moved to the left a distance λ. Therefore, the actual distance between crests emitted to the left will be

$$\lambda' = \lambda - u_s\tau_0.$$

These waves, having left the source, are of course moving at the speed of sound v relative to the air—the motion of the source does not affect the speed of sound in air. Therefore, as these waves of wavelength λ' arrive at an observer placed to the left so the source is moving directly towards him, he will hear a frequency $f' = v/\lambda'$.

That is, the observed frequency

$$f' = \frac{v}{\lambda'} = \frac{v}{\lambda - u_s\tau_0} = \frac{v}{\lambda}\left(\frac{1}{1 - u_s\tau_0/\lambda}\right) = f_0\left(\frac{1}{1 - u_s/v}\right).$$

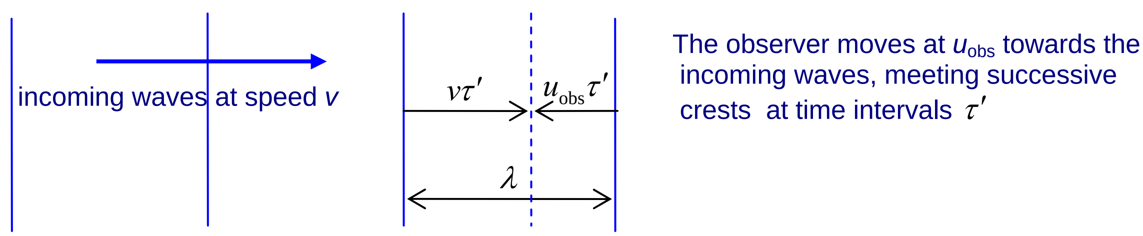
(Note that for the common case $(u_s/v) \ll 1$, we can approximate, $f' \cong f_0(1 + u_s/v)$.)

By an exactly parallel argument, for a source moving *away* from an observer at speed u_s, the frequency is lower by the corresponding factor:

For source moving away from observer $f' = f_0\left(\frac{1}{1 + u_s/v}\right)$.

Stationary Source, Moving Observer

Consider now an observer moving at speed u_{obs} directly towards a stationary frequency f_0 source. So, she's moving to meet the oncoming wave crests. Remember, the wave crests are λ apart in the air, and moving at v. Suppose her time between meeting successive crests is τ'. During this time, she moves $u_{obs}\tau'$, the wave crest moves $v\tau'$ coming to meet her, and between them they cover the distance λ between crests.



It is evident from the diagram that the time interval she will measure between meeting successive crests is

$$\tau' = \frac{\lambda}{u_{\text{obs}} + v}$$

and therefore the sound frequency she measures is

$$f' = \frac{1}{\tau'} = \frac{u_{\text{obs}} + v}{\lambda} = \frac{v}{\lambda}\left(1 + \frac{u_{\text{obs}}}{v}\right) = f_0\left(1 + \frac{u_{\text{obs}}}{v}\right).$$

Source and Observer Both Moving Towards Each Other

For this case, the arguments above can be combined to give:

$$f' = f_0\left(\frac{1 + u_{\text{obs}}/v}{1 - u_{\text{s}}/v}\right).$$

Both motions increase the observed frequency. If either observer or source is moving in the opposite direction, the observed frequency is found by switching the sign of the corresponding u.

Doppler Effect for Light

The argument above for the Doppler frequency shift is accurate for sound waves and water waves, but fails for light and other electromagnetic waves, since their speed is not relative to an underlying medium, but to the observer. To derive the Doppler shift in this case requires special relativity. A derivation can be found in my Modern Physics notes.

The Doppler shift for light depends on the relative velocity u of source and observer:

$$f' = f_0\sqrt{\frac{1 + u/c}{1 - u/c}}$$

for motion towards each other.

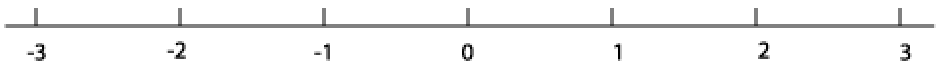
Other Possible Motions of Source and Observer

We've assumed above that the motions of source and observer are all along the same straight line. But as we hear the change in frequency of a jet engine passing overhead, the note drops smoothly, because we're off the straight line path of the plane. The actual note heard as a function of time can be found from fairly simple geometric considerations to be $f' = f_0/\left(1 - u_s \cos\theta / v\right)$, where θ is the angle between the straight line path and a line from the source to the observer. This factor is incorporated in police speed radar units. One interesting point: if $\theta = \pi/2, \quad f' = f_0$. This seems very reasonable, but is *not* the case for light, where observed time dilation of the source gives a frequency shift. This was found unequivocally in a beautiful series of experiments in the 1930's (by Ives and Stillwell) attempting to disprove special relativity.

Appendix: Complex Numbers

Real Numbers

Let us think of the ordinary numbers as set out on a line which goes to infinity in both positive and negative directions. We could start by taking a stretch of the line near the origin (that is, the point representing the number zero) and putting in the integers as follows:



Next, we could add in rational numbers, such as ½ , 23/11, etc., then the irrationals like $\sqrt{2}$, then numbers like π , and so on, so any number you can think of has its place on this line.

Now let's take a slightly different point of view, and think of the numbers as represented by a *vector* from the origin to that number, so 1 is

and, for example, –2 is represented by:

Note that if a number is multiplied by –1, the corresponding vector is turned through 180 degrees. In pictures,

The "vector" 2 is turned through π , or 180 degrees, when you multiply it by –1.

What are the square roots of 4?

Well, 2, obviously, but also –2, because multiplying the backwards pointing vector –2 by –2 not only doubles its length, but also turns it through 180 degrees, so it is now pointing in the positive direction. We seem to have invented a hard way of stating that multiplying two negatives gives a positive, but thinking in terms of turning vectors through 180 degrees will pay off soon.

Solving Quadratic Equations

In solving the standard quadratic equation

$$ax^2 + bx + c = 0$$

we find the solution to be:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The problem with this is that sometimes the expression inside the square root is negative. What does that signify? For some problems in physics, it means there is no solution. For example, if I throw a ball directly upwards at 10 meters per sec, and ask when will it reach a height of 20 meters, taking g = 10 m per sec^2, the solution of the quadratic equation for the time t has a negative number inside the square root, and that means that the ball doesn't get to 20 meters, so the question didn't really make sense.

We shall find, however, that there are other problems, in wide areas of physics, where negative numbers inside square roots have an important physical significance. For that reason, we need to come up with a scheme for interpreting them.

The simplest quadratic equation that gives trouble is:

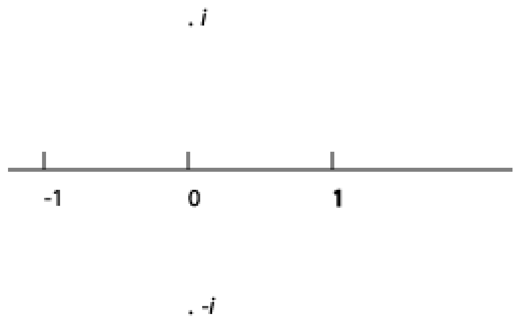
$$x^2 + 1 = 0$$

the solutions being $x = \pm\sqrt{-1}$.

What does that mean? We've just seen that the square of a positive number is positive, and the square of a negative number is also positive, since multiplying one negative number, which points backwards, by another, which turns any vector through 180 degrees, gives a positive vector. Another way of saying the same thing is to regard the minus sign itself, –, as an operator which turns the number it is applied to through 180 degrees. Now $(-2)\times(-2)$ has two such rotations in it, giving the full 360 degrees back to the positive axis.

To make sense of the square root of a negative number, we need to find something which when multiplied by itself gives a negative number. Let's concentrate for the moment on the square root of –1, from the quadratic equation above. Think of –1 as the operator – acting on the vector 1, so the – turns the vector through 180 degrees. We need to find the square root of this operator, the operator which applied *twice* gives the rotation through 180 degrees. Put like that, it is pretty obvious that the operator we want rotates the vector 1 through 90 degrees.

But if we take a positive number, such as 1, and rotate its vector through 90 degrees only, it isn't a number at all, at least in our original sense, since we put all known numbers on one line, and we've now rotated 1 away from that line. The new number created in this way is called a pure imaginary number, and is denoted by i.



Once we've found the square root of –1, we can use it to write the square root of any other negative number—for example, $2i$ is the square root of –4. Putting together a real number from the original line with an imaginary number (a multiple of i) gives a *complex number*. Evidently, complex numbers fill the entire two-dimensional plane. Taking ordinary Cartesian coordinates, any point P in the plane can be written as (x, y) where the point is reached from the origin by going x units in the direction of the positive real axis, then y units in the direction defined by i, in other words, the y axis.

Thus the point P with coordinates (x, y) can be identified with the complex number z, where

$$z = x + iy.$$

The plane is often called the *complex plane*, and representing complex numbers in this way is sometimes referred to as an Argand Diagram.

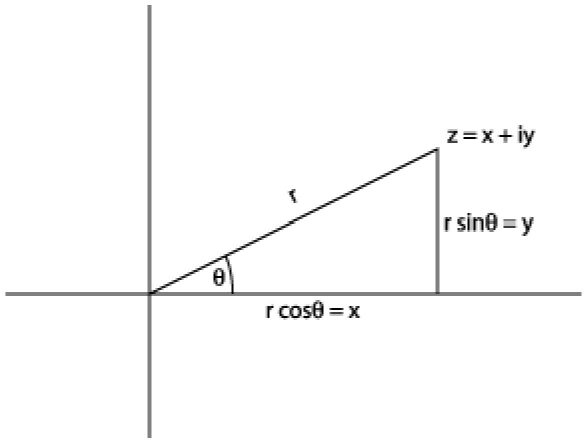
Visualizing the complex numbers as two-dimensional vectors, it is clear how to *add* two of them together. If $z_1 = x_1 + iy_1$, and $z_2 = x_2 + iy_2$, then $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$. The real parts and imaginary parts are added separately, just like vector components.

Multiplying two complex numbers together does not have quite such a simple interpretation. It is, however, quite straightforward—ordinary algebraic rules apply, with i^2 replaced where it appears by –1. So for example, to multiply $z_1 = x_1 + iy_1$ by $z_2 = x_2 + iy_2$,

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

Polar Coordinates

Some properties of complex numbers are most easily understood if they are represented by using the polar coordinates (r,θ) instead of (x, y) to locate z in the complex plane.



Note that $z = x + iy$ can be written $r(\cos\theta + i\sin\theta)$ from the diagram above. In fact, this representation leads to a clearer picture of multiplication of two complex numbers:

$$\begin{aligned} z_1 z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2 \{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\} \\ &= r_1 r_2 (\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)). \end{aligned}$$

So, if

$$z = r(\cos\theta + i\sin\theta) = z_1 z_2,$$

then

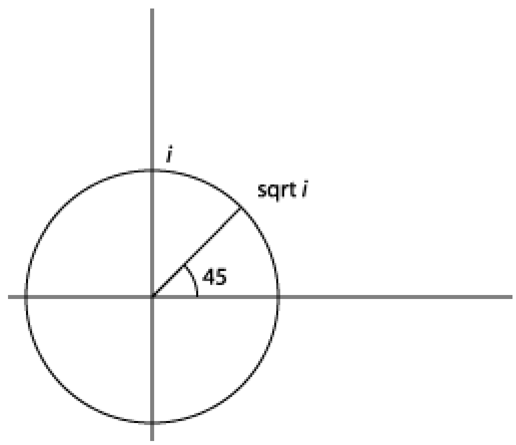
$$r = r_1 r_2$$

and

$$\theta = \theta_1 + \theta_2.$$

That is to say, to multiply together two complex numbers, we *multiply* the r's – called the *moduli* – and *add* the phases, the θ 's. The modulus r is often denoted by $|z|$, and called mod z, the phase θ is sometimes referred to as arg z. For example, $|i| = 1$, arg $i = \pi/2$.

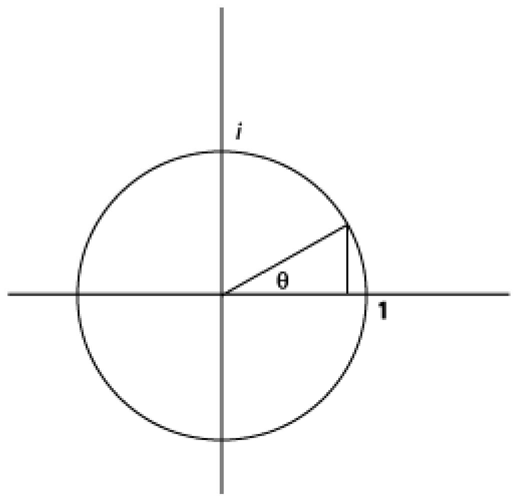
We can now see that, although we had to introduce these complex numbers to have a $\sqrt{-1}$, we don't need to bring in new types of numbers to get $\sqrt{-i}$, or $\sqrt{i}$. Clearly, $\left|\sqrt{i}\right| = 1$, arg $\sqrt{i} = 45°$. It is on the circle of unit radius centered at the origin, at 45°, and squaring it just doubles the angle.



The Unit Circle

In fact this circle—called the *unit circle*—plays an important part in the theory of complex numbers. Every point on the circle has the form

$$z = \cos\theta + i\sin\theta = \text{Cis}(\theta), \text{ say.}$$



Since all points on the unit circle have $|z| = 1$, by definition, multiplying any two of them together just amounts to adding the angles, so our new function $\text{Cis}(\theta)$ satisfies

$$\text{Cis}(\theta_1)\text{Cis}(\theta_2) = \text{Cis}(\theta_1 + \theta_2).$$

But that is just how multiplication works for exponents!

That is, $a^{\theta_1}a^{\theta_2} = a^{\theta_1+\theta_2}$ for a any constant, which strongly suggests that maybe our function $\text{Cis}(\theta)$ is nothing but some constant a raised to the power θ, that is, $\text{Cis}(\theta) = a^{\theta}$.

It turns out to be convenient to write $a^{\theta} = e^{(\ln a)\theta} = e^{A\theta}$, say, where $A = \ln a$.

This line of reasoning leads us to write $\cos\theta + i\sin\theta = e^{A\theta}$.

Now, for the above "addition formula" to work for multiplication, A must be a constant, *independent* of θ. Therefore, we can find the value of A by choosing θ for which things are simple. We take θ to be very small—in this limit $\cos\theta = 1$, $\sin\theta = \theta$, and $e^{A\theta} = 1 + A\theta$, dropping terms of order θ^2 and higher.

Substituting these values into $\cos\theta + i\sin\theta = e^{A\theta}$ gives $A = i$.

So we find:

$$\boldsymbol{(\cos\theta + i\sin\theta) = re^{i\theta}}.$$

To test this result, we expand $e^{i\theta}$:

$$\begin{aligned}
e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \ldots \\
&= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \ldots \\
&= (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots) \\
&= \cos\theta + i\sin\theta
\end{aligned}$$

We write $= \cos\theta + i\sin\theta$ in the last line because the series in the brackets are precisely the Taylor series for $\cos\theta$ and $\sin\theta$, confirming our equation for $e^{i\theta}$. Changing the sign of θ it is easy to see that

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

so

$$\cos\theta = \tfrac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right), \text{ and } \; i\sin\theta = \tfrac{1}{2}\left(e^{i\theta} - e^{-i\theta}\right).$$

<u>Bottom Line</u>: any complex number can be written:

$$\boldsymbol{z = r(\cos\theta + i\sin\theta) = re^{i\theta}}$$

Complex Exercises

1. Show where in the complex plane are: 1, i, $-1+\sqrt{3}i, \sqrt{i}, \sqrt{\sqrt{i}}$, and write all these numbers in the form $re^{i\theta}$.
2. State the rule for multiplying two complex numbers of the form $re^{i\theta}$, and from that figure out the *inverse* of a complex number: that is, express $1/\left(re^{i\theta}\right)$ as $r_1 e^{i\theta_1}$.

3. Find how to invert a number in the other notation: if $\dfrac{1}{x+iy} = a+ib$, find a, b in terms of x, y.

Hint: it helps to multiply $\dfrac{1}{x+iy}$ by $\dfrac{x-iy}{x-iy}$.

4. Show on a diagram where in the complex plane is a *cube* root of -1, we'll call it ω. How many cube roots does -1 have? Show all possibilities on the diagram. Next, what about cube roots of 1? Show them on another figure. (*Note*: ω is commonly used for a cube root of -1. we also use it, of course, for angular frequency. Take care not to confuse the two.)

5. Draw a complex number z as a vector (pointing from the origin to z), then draw on the same diagram as vectors iz, z/i, ωz. (ω being the cube root of -1.)

6. Using $e^{i\theta} = \cos\theta + i\sin\theta$, from $e^{i(A+B)} = e^{iA}e^{iB}$, deduce the formulas for $\sin(A+B), \cos(A+B)$.

7. Suppose the point z moves in the complex plane is such a way that at time t $z(t) = Ae^{i\omega_0 t}$, where A is real and $\omega_0 = 2\pi$ sec^{-1}. Where is z at t = 0? Where at t = 1 second? Where at t = 0.5 seconds? Where at t = 0.25 seconds? Describe how z moves as time progresses.

How would your answer change if A were *pure imaginary* instead of real?

8. Consider again $z(t) = Ae^{i\omega_0 t}$, $\omega_0 = 2\pi$ sec^{-1}. Differentiate both sides to find an expression for the velocity $\dot{z}(t) = dz/dt$ as the point moves along its path. How does the velocity vector relate to the position vector? Next, find by differentiating again the acceleration vector, and comment on its direction.

9. State briefly how z behaves in time if $z(t) = Ae^{i\omega t}$ for real ω. How would this behavior change if ω had a small imaginary part, $\omega = \omega_0 + i\Gamma$, where Γ is small? Sketch how z would move in the complex plane, both for Γ positive *and* Γ negative.

10. Consider the quadratic equation $x^2 - 2bx + 1 = 0$. For b = 1, both roots equal 1. Sketch (in the complex plane) how the larger root moves as b varies from 1.2 down through 1 to 0.8. When you've done that, do the same for the other root, preferably in a different color.

Oscillations and Waves Homework Problems

Oscillations

1. *Dimensional exercises*: use dimensions to find a characteristic time for an undamped simple harmonic oscillator, and a pendulum. Why does the dimensional argument work for any initial displacement of the oscillator, but only small swings of the pendulum?

What possible characteristic times can be found dimensionally for a damped oscillator? Explain the physical significance of these times for a heavily damped oscillator, and a lightly damped oscillator.

2. (a) A *heavily damped* oscillator has mass m, spring constant k and damping force $-bv$, where $v = dx/dt$.

Before $t = 0$, the mass is at rest at $x = 0$, but at $t = 0$, a sudden kick gives it velocity v_0.

Sketch a graph of how v varies in time after that. (You are not expected to solve the equation here, just sketch the behavior.) Is the behavior immediately after the kick any different from the behavior later on? Are all the parameters m, k, b equally important throughout the motion? Explain briefly.

(b) Suppose the heavily damped oscillator is pulled to x_0 away from $x = 0$, then released from rest. Sketch its position as a function of time. State which of m, k, b are important immediately after the mass is released, and which are important later on.

3. Open the damped oscillator spreadsheet. Let's first examine damped motion *without* the spring. Set $m = 1$, $k = 0$, $b = 3$, $x_{\text{init}} = 0$, $v_{\text{init}} = 3$.

(a) How does the curve relate to the dimensionally derived time(s) for a damped oscillator?

(b) Write down the equation for this damped $k = 0$ "oscillator". (Of course, this won't oscillate!). Put $dx/dt = v$, to get a first-order equation for v. Solve it, and see if your solution agrees with the spreadsheet curve.

(c) Now you've found $v(t)$, you know dx/dt. Write down and solve the equation for $x(t)$, and check that it agrees with the spreadsheet.

(d) Bring back the spring: set $k = 1$. Does this significantly change the initial shooting upwards of the curve? What, then, are the important terms in the equation for that initial part of the motion?

(e) Look at the very top of the curve, the maximum value of x: what are the important terms in the equation in that neighborhood?

(f) For longer times, which terms in the equation dominate? Drop the least important term, and solve the remaining equation. Then check to see if this is a good approximation or not.

4. Open the damped oscillator spreadsheet. Fix the constants to give the critical damping curve. Then lower the damping *b* until you can detect the oscillator going past the origin.

(a) Roughly, by what percentage do you need to lower *b* to see this?

(b) Suppose in building a model for a shock absorber you were willing to let the downward swing be as much as 5% of the original upward displacement, and you take $m = 1$, $k = 1$ for the model, what would be the value of *b*?

5. Open the damped driven oscillator spreadsheet and put $k = 1$, $b = 0.1$, omega = 1.25, delta_t = 0.055.

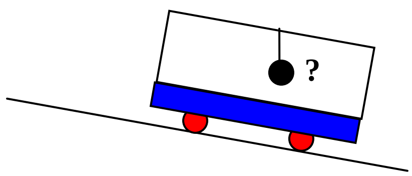
(a) What's going on at the beginning? It might help to set $b = 0$ temporarily to get some insight.

(b) Note that the solution settles down to a steady state. Does the time to settle down depend on the initial conditions? Change them and find out. Set x_{init} large, for example. Can you arrange the initial conditions so that the steady solution takes over immediately? How would you do that?

6. (a) Open the pendulum spreadsheet. See how the period varies with the amplitude: it's initially set at 0.1 radian. Try 1 radian, 2 radians, 3 radians.

(b) In the pendulum spreadsheet, set the initial angle theta_init =0, then try the initial angular velocity omega_init = 4, 5, 6, 7. Interpret your result. How can you make the pendulum period very long?

7. An unpowered streetcar is accelerating under gravity down a ten degree slope. Neglect friction and air resistance, assume the acceleration is the same as a smooth block sliding down a frictionless surface. A pendulum of length *l* is hung from the ceiling inside.



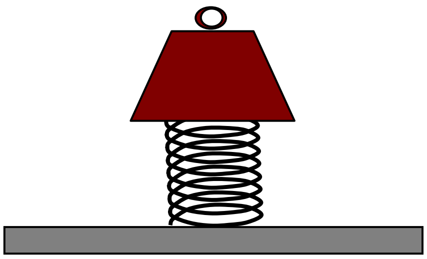
(a) If the pendulum is at rest, what is the direction of the string?

(b) What is the period of oscillation of the pendulum?

8. (a) Imagine a tunnel can be drilled directly through the center of the Earth, insulated from internal heat, and the air evacuated. You drop a package into the tunnel. How long does it take to reach the surface at the other end? How does that time compare with a low orbit satellite?

(b) Suppose a tunnel is constructed *in a straight line* from New York to Los Angeles, so it doesn't follow the Earth's curvature. Estimate how steep the downhill gradient is at the ends. If a frictionless maglev train goes through the tunnel in a vacuum, with no power but gravity, how long will the trip take?

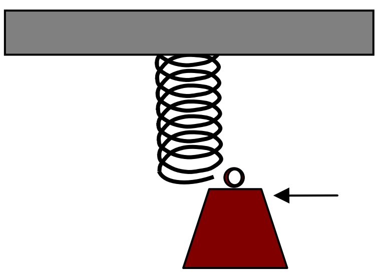
9. A 1 kg mass rests on a spring. A gentle downward pulse causes vertical oscillations at 5Hz.



(a) Suppose a balloon is attached to the top of the mass. The balloon has a mass of 0.05kg, but feels a buoyancy force able to lift 0.55 kg. How does this affect the period of the oscillation?

(b) What would be the period of the mass + spring (no balloon) on the Moon? (g_{Moon} = 2 m/s^2.)

10. A spring is hanging vertically at rest. A mass held in the hand is gently attached to the end of the spring, then released. The system oscillates, the maximum downward distance being 3 cm below the original position.



What is the period of oscillation?

11. Let us represent a ship weighing 20,000 tons (1 ton = 1,000 kg) by a rectangular parallelepiped, 150 m long, 30 m across, 20 m deep. Show that in vertical motion, this ship behaves as a simple harmonic oscillator, and find the period.

12. A flat horizontal plate driven from below oscillates vertically with an amplitude of 1 mm. Some sand (of negligible mass) is sprinkled on the plate. The frequency of the oscillator is gradually increased from zero. At what frequency will the sand lose contact with the plate? At what point in the cycle will this happen?

13. (a) Prove that for a lightly damped oscillator, the change in frequency caused by the damping is approximately $\omega_0 / 8Q^2$.

(b) If damping causes a 1% decrease in the frequency of an oscillator, what is its Q value? Over how many cycles does the energy drop by $1/e$? Over how many cycles does the energy drop by ½?

14. A lightly damped oscillator has mass m, spring constant k and damping factor b.

(a) Prove that at any instant of time the rate of loss of energy is bv^2 joules/sec., v being the instantaneous velocity.

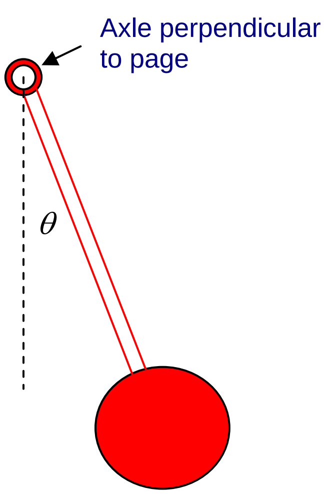
(b) Assuming the change in amplitude in a single cycle is negligible, what is the *average* value of v^2 over the cycle compared with the *maximum* value of v^2 ?

(c) If he energy loss in one second is small, show that it is well approximated by $E(t=1)=E(t=0)(1-b\times 1/m)$, and deduce that for long times the energy decays as $e^{-bt/m}$.

15. A lightly damped driven oscillator exhibits a strong resonance at frequency ω_0. Prove that at resonance, the total energy in the oscillator for a given driving force is proportional to Q^2.

16. An old but precisely made pendulum clock keeps time within one second a day in Charlottesville. The proud owner takes it to a new apartment in Wintergreen, about 3000 feet above Charlottesville in altitude. If the clock is not adjusted, how many seconds a day will it gain or lose? (Assume the new room location is kept at the same temperature as the earlier place.)

17. The bob of a pendulum is a uniform disk of radius 4 cm , attached to the end of a very light rod, so that the center of the bob is a distance 30 cm from the support axle (which would be perpendicular to the page).



(a) Find the moment of inertia of the bob about the axle (you'll need to use the Parallel Axis Theorem).

(b) For small oscillations, what percentage error in the period arises in using the simple point-mass approximation?

18. (a) A pendulum clock that keeps perfect time in the ground floor lobby of the Empire State Building is taken to a room at the top of the building, 1250 feet high. How many seconds (or what fraction of a second) do you expect it to gain or lose per day? Assume the temperature is the same, and neglect the mass of the building.

(b) Actually, the building weighs 300,000 tons. Figure out if that will affect your estimate significantly.

19. In the lecture notes on oscillations, we wrote the equation of motion of the driven damped oscillator as

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=F_0\cos\omega t$$

and solved it by replacing $F_0\cos\omega t$ with $F_0e^{i\omega t}$: the equation is then quite easy to solve, putting in a trial complex number solution $z(t)=Ae^{i(\omega t+\varphi)}$ (having real part $x(t)=A\cos(\omega t+\varphi)$) we found expressions for A and ϕ.

(a) Derive the expressions for A and ϕ. Show on a complex plane diagram $F_0e^{i\omega t}$ and $z(t)$ at some instant in time, and also show a vector (or complex number) representing the velocity dz/dt at that instant. State how these complex numbers move as time goes on.

(b) Check that the real part $x(t)=A\cos(\omega t+\varphi)$ is in fact a solution of the differential equation.

(c) Sketch how A varies with ω, especially near ω_0, for small damping b.

(d) Sketch how φ varies with ω, especially near ω_0, for small damping b.

(e) For the real solution, when the oscillator moves through Δx, the driving force does work $\left(F_0\cos\omega t\right)\Delta x$. Prove from this that the rate of working of the driving force is $\left(F_0\cos\omega t\right)v\left(t\right)$, where $v(t)$ is the velocity at time t. By averaging over a complete cycle, find the average rate of working – the power – of the driving force.

(f) How does the power vary with ω, especially near ω_0, for small damping b? Give a brief explanation.

20. We proved in the lecture that the steady-state solution for the damped oscillator driven by a force

$$F\left(t\right)=F_0\cos\omega t$$

is

$$x\left(t\right)=A\cos\left(\omega t-\theta\right),\quad \text{where}$$

$$A=\frac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2+(b\omega)^2}},$$

$$\tan\theta=\frac{b\omega}{m\left(\omega_0^2-\omega^2\right)}.$$

(a) Prove that the total energy in the oscillator, kinetic + potential, usually varies through the cycle. (Note that ω itself does not vary.)

Compare the rate of working of the driving force and that of the damping force, and explain how your result ties in with the answer to part (a).

(b) Prove that at the resonant frequency, the energy in the oscillator is

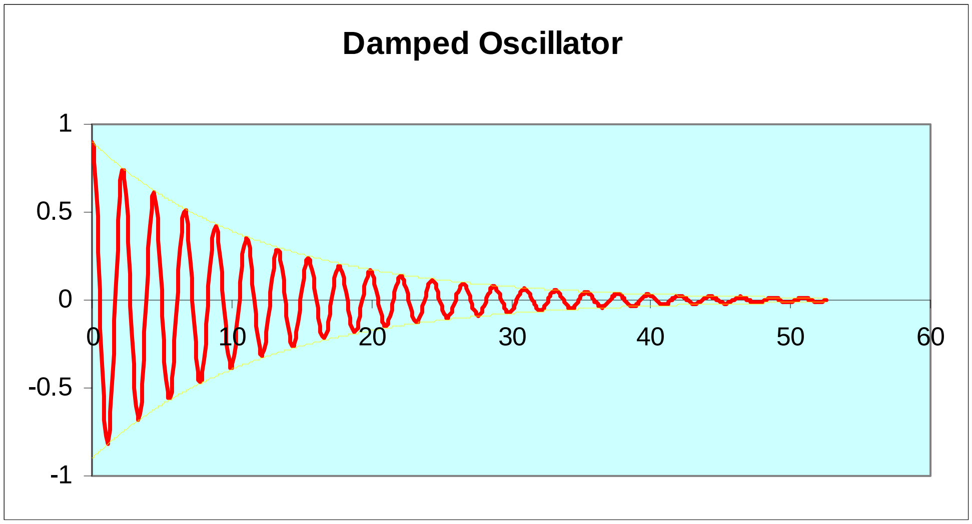
$$E_{\text{resonance}} = \frac{Q^2}{2}\frac{F_0^2}{m\omega_0^2}.$$

(c) Prove that the power input (rate of working) of the driver at resonance is

$$P_{\text{resonance}} = \frac{Q}{2}\frac{F_0^2}{m\omega_0}.$$

(d) The power input will drop to half on varying ω away from resonance when the denominator $m^2(\omega_0^2-\omega^2)^2+(b\omega)^2$ doubles. Assume Q is large, so you can replace $b\omega$ by $b\omega_0$ over this range, and conclude the power input is halved at $\omega-\omega_0 \cong \pm\omega_0/2Q$. Sketch very roughly the power input as a function of driving frequency for a large Q, then for *double* that Q on the same sketch.

21. (a) Estimate Q for the following oscillator (and don't forget the energy is proportional to the *square* of the amplitude):



(b) What kind of Q value would you expect for a guitar string? Is a high Q value in a musical instrument always a good thing?

Waves

22. Sketch the appearance of a single transverse wave pulse traveling down a string at some instant, and below it sketch the velocities of the small segments of string at that same instant.

23. (a) Write down the wave equation for a string mass μ per unit length, tension T.

(b) With the aid of a diagram, explain briefly how the wave equation follows from $F = ma$. (You don't need to put in mathematical details, just make it sound plausible.)

(c) Prove that any function of the form $f(x - vt)$ *or* $f(x + vt)$ satisfies the equation.

(d) Show how a standing harmonic wave can be constructed by adding two waves traveling in opposite directions.

24. (a) For a standing harmonic wave on a string, draw the position of the string when the kinetic energy is a maximum.

(b) For a standing wave having transverse displacement $y(x, t) = A\sin kx\cos\omega t$, what is the *total* energy in one wavelength?

25. (a) For a sound wave in air, the velocity depends on the density of the air ρ and the bulk modulus B. Use the method of dimensions to find v (apart from a dimensionless constant) as a function of ρ and B.

(b) In a sound wave, how does the air density variation at any point relate to the displacement $s(x, t)$? State the result, and then draw a simple diagram with s varying as a function of x at some instant of time t to make your answer plausible.

(c) In class, we showed that a tuning fork oscillating at 512 cycles per second caused resonance in a tube closed at one end, open at the other, of length about 16cm, and there were no resonances at shorter lengths. From this observation, figure out the speed of sound – but you must justify any statements about nodes, antinodes, etc., in picturing the resonating gas in the tube.

26. Assume a wave $y = A\sin(kx - \omega t)$ is traveling down a long taut string.

(a) Sketch the form of the wave at $t = 0$, showing the wavelength λ on your diagram.
(b) What is λ in terms of k?
(c) At $x = 0$, the string oscillates at frequency f Hz (cycles per second). What is f in terms of ω?
(d) What is the speed of the wave in terms of λ and f? Give a short explanation of your result.
(e) In what way is the wave $y = A\sin(kx + \omega t)$ different from $y = A\sin(kx - \omega t)$?

(f) Suppose the two waves in (e) are added. Draw a diagram of the resulting wave form, and describe how it varies in time.

27. Suppose that in a standing sound wave in the air in an organ pipe, the displacement of the air at point x at time t is $A\sin kx\cos\omega t$.

(a) Draw a sketch of this displacement curve at $t = 0$, and show on your curve where the *pressure* is a maximum.

(b) Give a brief justification for your choice of where the pressure is a maximum in your answer to (a).

(c) Consider an organ pipe of length L, closed at one end and open at the other. What condition does the displacement satisfy at the *closed* end of the pipe? Explain briefly.

(d) Same as (c), but at the *open* end of the pipe.

(e) Give an approximate value for the length of the organ pipe if the lowest note is 34 Hz.

(f) What is the next lowest note (resonant frequency) for this pipe?

28. A uniform wire one meter long is held at tension 500N. It has a mass of 0.05kg. It is vibrating in its fundamental mode with an amplitude of 0.5cm.

(a) What is its maximum kinetic energy? What is its instantaneous shape at the moment of maximum kinetic energy?

(b) What is its maximum potential energy? What is its instantaneous shape at the moment of maximum potential energy?

(c) How are your answers to (a), (b) changed if the wire is vibrating in its first harmonic with the same amplitude?

29. Given that the note middle C corresponds to a frequency of 262 Hz., find the length of an organ pipe having this as its fundamental (lowest frequency) note
(a) if both ends of the pipe are open
(b) if one end is closed, one end open.
(c) What note would the organ pipe sound if filled with helium gas? (the bulk modulus is the same as for air).

30. Humans can only hear sounds in the frequency range 20 Hz to 20,000 Hz. What would be the longest and shortest organ pipes there is any point in manufacturing?

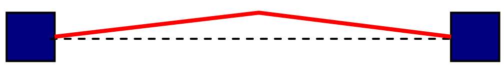
31. The speed of sound in water is about 1.5 km/sec.
(a) Figure out from that the bulk modulus of water.

(b) How compressed in volume is the water at the bottom of the deepest ocean?

32. A piston (a speaker) at one end of a long tube is oscillating at 262 Hz with an amplitude of 0.2mm. The piston is circular, with a diameter 5 cm.
(a) Find the maximum pressure variation at the plate.
(b) What is the average power output of the plate?
(c) How much energy is there in one meter of the traveling wave as it goes down the tube?

33. Assume a steel piano wire is 50 cm long and weighs 3 grams. It is held at a tension of 1,000N. (a) What is its fundamental frequency?
(b) What is the wavelength of that sound in air?
(c) How does the speed of a traveling wave in the steel piano wire compare to the speed of sound in air? The speed of sound in solid steel is far greater than the speed of sound in air. Does that contradict your finding for the wire? Explain why or why not.

34. A string under tension is pulled aside in the middle so it has the following V-shape:



Describe with a series of sketches the subsequent motion of the string.
(*Hint*: study the spreadsheet addition of two traveling sine waves to form a standing wave. The initial string configuration here is half a wave length of a zigzag wave.)

35. A police radar unit operates at a frequency of 24 GHz. A car is approaching such a unit at 80 mph. What is the beat frequency the unit detects between the emitted signal and the echo from the car?

36. A tsunami is a wave typically of height of order one meter, and wavelength of order one hundred kilometers—much greater than the ocean depth. This is called a wave in "shallow water", and in this case it is found that the wave speed does not depend on the wavelength, but only on the ocean depth and g.

(a) Use dimension arguments to find an order of magnitude estimate for the speed of a tsunami wave in the open ocean.

(b) How does the wave speed change on approaching shore? Give a ballpark estimate. Assuming there is only a small loss of energy, how will the height of the wave change as it approaches the shore?

37. Open the spreadsheet
http://galileo.phys.virginia.edu/classes/152.mf1i.spring02/WaveSum.xls from the Web Notes page. Use it to plot the sum of two waves close in wavelength and frequency to get beats. (Hint: you can change the scale with delta_x to get a wider view.) How does the frequency of the beats relate to the frequencies of the two waves beating together?

For some waves, such as those in water, the wave speed changes with wave length. It's always true that $\omega = vk$, but v now depends on k. Take two waves with k's close together, but $\omega_1 / k_1 \neq \omega_2 / k_2$ (although close). Hold down the slider bar to see a movie of how the wave develops, and describe how it differs from that for $\omega_1 / k_1 = \omega_2 / k_2$.

38. A narrow pipe 1 meter long is open at both ends. The lowest frequency sound mode in the pipe is excited with a tuning fork.

(a) What is the wavelength of this lowest mode?

(b) What is the frequency of the tuning fork?

(c) Explain with a sketch and simple graphs how the displacement of the air in the pipe $s(x,t)$ varies with time at one end and in the middle.

(d) Explain similarly how the *pressure* varies with time at one end and in the middle, and comment on how the pressure variation correlates with the displacement variation.

39. (a) Draw a diagram explaining Young's two-slit experiment, and show with your diagram how the wavelength of light can be determined by observing the location of the first bright spot away from the center.

(b) From your diagram, derive the formula for the wavelength λ in terms of slit separation d, distance to screen D and distance between bright spots x. Mention any approximations you make.

(c) Explain how the pattern of bright spots changes if more slits are added, the distance between neighboring slits still being the same d.