

[previous](#) [index](#) [next](#)

Oscillations

Michael Fowler 3/24/07

Introduction

In this lecture, we will be looking at a wide variety of oscillatory phenomena. After a brief recap of undamped simple harmonic motion, we go on to look at a heavily damped oscillator. We do that before considering the lightly damped oscillator because the mathematics is a little more straightforward—for the heavily damped case, we don't need to use complex numbers. But they arise very naturally in the lightly damped case, and are great for understanding the *driven* oscillator and resonance phenomena, as will become apparent in later sections.

Brief Review of Undamped Simple Harmonic Motion

Our basic model simple harmonic oscillator is a mass m moving back and forth along a line on a smooth horizontal surface, connected to an inline horizontal spring, having spring constant k , the other end of the string being attached to a wall. The spring exerts a restoring force equal to $-kx$ on the mass when it is a distance x from the equilibrium point. By “equilibrium point” we mean the point corresponding to the spring resting at its natural length, and therefore exerting no force on the mass. The in-class realization of this model was an aircar, with a light spring above the track (actually, we used *two* light springs, going in opposite directions—we found if we just one it tended to sag on to the track when it was slack, but two in opposite directions could be kept taut. The two springs together act like a single spring having spring constant the sum of the two).

Newton's Law gives:

$$F = ma, \text{ or } m \frac{d^2x}{dt^2} = -kx.$$

Solving this differential equation gives the position of the mass (the aircar) relative to the rest position as a function of time:

$$x(t) = A \cos(\omega_0 t + \varphi).$$

Here A is the maximum displacement, and is called the **amplitude** of the motion. $\omega_0 t + \varphi$ is called the **phase**. φ is called the **phase constant**: it depends on where in the cycle you start, that is, where is the oscillator at time zero.

The velocity and acceleration are given by differentiating $x(t)$ once and twice:

$$v(t) = \frac{dx}{dt} = -A\omega_0 \sin(\omega_0 t + \varphi)$$

and

$$a(t) = \frac{d^2 x(t)}{dt^2} = -A\omega_0^2 \cos(\omega_0 t + \varphi).$$

We see immediately that this $x(t)$ does indeed satisfy Newton's Law provided ω_0 is given by

$$\omega_0 = \sqrt{k/m}.$$

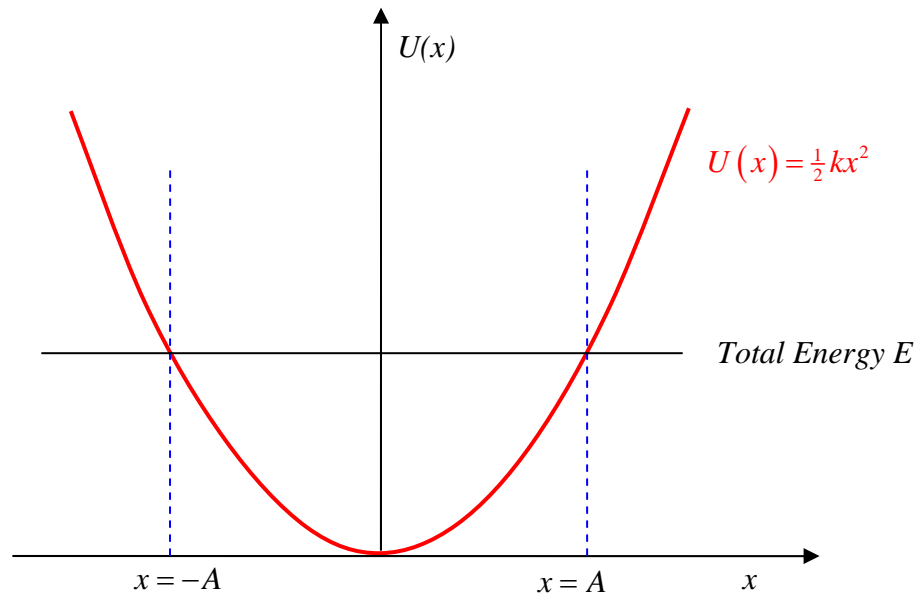
Exercise: Verify that, apart from a possible overall constant, this expression for ω_0 could have been figured out using dimensions.

Energy

The spring stores *potential energy*: if you push one end of the spring from some positive extension x to $x + dx$ (with the other end of the spring fixed, of course) the force $-kx$ opposes the motion, so you must push with force $+kx$, and therefore do work $kx dx$. To find the *total* potential energy stored by the spring when the end is x_0 away from the equilibrium point (natural length) we must find the total work required to stretch the spring from its natural length to an extension x_0 . This means adding up all the little bits of work $kx dx$ needed to get the spring from no extension at all to an extension of x_0 . In other words, we need to do an integral to find the potential energy $U(x_0)$:

$$U(x_0) = \int_0^{x_0} kx dx = \frac{1}{2} kx_0^2.$$

So the potential energy plotted as a function of distance from equilibrium is parabolic:



Potential Energy $U(x)$ for a Simple Harmonic Oscillator.

For **total** energy E , the oscillator swings back and forth between $x = -A$ and $x = +A$.

The oscillator has *total* energy equal to kinetic energy + potential energy,

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

when the mass is at position x . Putting in the values of $x(t)$, $v(t)$ from the equations above, it is easy to check that E is independent of time and equal to $\frac{1}{2}kA^2$, A being the amplitude of the motion, the maximum displacement. Of course, when the oscillator is at A , it is momentarily at rest, so has no kinetic energy.

A Heavily Damped Oscillator

Suppose now the motion is damped, with a drag force proportional to velocity. The equation of motion becomes:

$$m \frac{d^2x}{dt^2} = -kx - b \frac{dx}{dt}.$$

Although this equation looks more difficult, it really isn't! The important point is that the terms are just *derivatives* of x with respect to time, multiplied by *constants*. It would be a lot more difficult if we had a drag force proportional to the square of the velocity, or if the force exerted by the spring were not a constant times x (this means we can't stretch the string too far!). Anyway, it is easy to find exponential functions that are solutions to this equation. Let us guess a solution:

$$x = x_0 e^{-\alpha t}.$$

Inserting this in the equation, using

$$\frac{dx}{dt} = -\alpha x_0 e^{-\alpha t}, \quad \frac{d^2x}{dt^2} = \alpha^2 x_0 e^{-\alpha t}$$

we find that it *is* a solution provided that α satisfies:

$$m\alpha^2 - \alpha b + k = 0$$

from which

$$\alpha = \frac{b \pm \sqrt{b^2 - 4mk}}{2m}.$$

Staring at this expression for α , we notice that for α to be real, we need to have

$$b^2 > 4mk.$$

What can that mean? Remember b is the damping parameter—we're finding that our proposed exponential solution only works for *large* damping! Let's analyze the large damping case now, then after that we'll go on to see how to extend the solution to small damping.

Interpreting the Two Different Exponential Solutions

It's worth looking at the case of *very* large damping, where the two exponential solutions turn out to decay at very different rates. For b^2 *much* greater than $4mk$, we can write

$$\alpha = \frac{b \pm b \sqrt{1 - \frac{4mk}{b^2}}}{2m} = \alpha_1, \alpha_2$$

and then expand the square root using

$$(1 - x)^{1/2} \cong 1 - \frac{1}{2}x,$$

valid for small x , to find that approximately—for large b —the two possible values of α are:

$$\alpha_1 = \frac{b}{m} \quad \text{and} \quad \alpha_2 = \frac{k}{b}.$$

That is to say, there are two possible highly damped decay modes,

$$x = A_1 e^{-\alpha_1 t} \quad \text{and} \quad x = A_2 e^{-\alpha_2 t}.$$

Note that since the damping b is large, α_1 is *large*, meaning *fast* decay, and α_2 is *small*, meaning *slow* decay.

Question: what, physically, is going on in these two different highly damped exponential decays? Can you construct a plausible scenario of a mass on a spring, all in molasses, to see why two very different rates of change of speed are possible?

Hint: look again at the equation of motion of this damped oscillator. Notice that in each of these highly damped decays, one term doesn't play any part—but the irrelevant term is a *different* term for the two decays!

Answer 1: for $\alpha = k/b$, evidently the *mass* doesn't play a role. This decay is what you get if you pull the mass to one side, let go, then, after it gets moving, it will very slowly settle towards the equilibrium point. Its rate of approach is determined by balancing the spring's force against the speed-dependent damping force, to give the speed. The rate of change of speed—the acceleration—is so tiny that the *inertial* term—the mass—is negligible.

Answer 2: for $\alpha = b/m$, the *spring* is negligible. And, this is very *fast* motion ($b/m \gg k/b$, since we said $b^2 \gg 4mk$.) The way to get this motion is to pull the mass to one side, then give it a very strong kick towards the equilibrium point. If you give it just the right (high) speed, all the momentum you imparted will be spent overcoming the damping force as the mass moves to the center—the force of the spring will be negligible.

*The Most General Solution for the Highly Damped Oscillator

The damped oscillator equation

$$m \frac{d^2 x}{dt^2} = -kx - b \frac{dx}{dt}$$

is a linear equation. This means that if $x_1(t)$ is a solution, and $x_2(t)$ is another solution, that is,

$$m \frac{d^2 x_1(t)}{dt^2} = -kx_1(t) - b \frac{dx_1(t)}{dt}$$

$$m \frac{d^2 x_2(t)}{dt^2} = -kx_2(t) - b \frac{dx_2(t)}{dt}$$

then just adding the two equations we get:

$$m \frac{d^2 (x_1(t) + x_2(t))}{dt^2} = -k(x_1(t) + x_2(t)) - b \frac{d(x_1(t) + x_2(t))}{dt}.$$

It is also clear that *multiplying a solution by a constant produces another solution*: if $x(t)$ satisfies the equation, so does $3x(t)$.

This means, then, that given two solutions $x_1(t)$ and $x_2(t)$, and two arbitrary constants A_1 and A_2 , the function

$$A_1 x_1(t) + A_2 x_2(t)$$

is also a solution of the differential equation.

In fact, all possible motions of the highly damped oscillator have this form. The way to understand this is to realize that the oscillator's motion is *completely determined* if we specify at an initial instant of time *both* the position *and* the velocity of the oscillator. The equation of motion gives the acceleration as a function of position and velocity, so, at least in principle, we can work out step by step how the mass must move; technically, we are integrating the equation of motion, either mathematically, or numerically such as by using a spreadsheet. So, by suitably adjusting the two arbitrary constants A_1 and A_2 , we can match our sum of solutions to any given initial position and velocity.

To summarize, for the highly damped oscillator any solution is of the form:

$$x(t) = A_1 e^{-\alpha_1 t} + A_2 e^{-\alpha_2 t} = A_1 e^{-\frac{b+b\sqrt{1-\frac{4mk}{b^2}}}{2m}t} + A_2 e^{-\frac{b-b\sqrt{1-\frac{4mk}{b^2}}}{2m}t}.$$

Exercises on highly damped oscillations

1. If the oscillator is pulled aside a distance x_0 , and released from rest at $t = 0$, what are A_1, A_2 ? Describe the subsequent motion, especially the very beginning: what is the initial acceleration? (*Hint*: think carefully about how important the damping term is immediately after release from rest—you should be able to *guess* the initial acceleration.)

2. If the oscillator is initially at the equilibrium position $x_0 = 0$, but is given a kick to a velocity v_0 , find A_1 and A_2 and describe the subsequent motion.

*The Principle of Superposition for Linear Differential Equations

The equation for the highly damped oscillator is a linear differential equation, that is, an equation of the form (in more usual notation):

$$c_0 f(x) + c_1 \frac{df(x)}{dx} + c_2 \frac{d^2 f(x)}{dx^2} = 0$$

where c_0, c_1 and c_2 are constants, that is, independent of x .

For such a linear differential equation, if $f_1(x)$ and $f_2(x)$ are solutions, so is $A_1 f_1(x) + A_2 f_2(x)$ for any constants A_1, A_2 . This is called the **Principle of Superposition**, and is proved in general *exactly* as we proved it for the highly damped oscillator in the preceding section.

Even more important, this Principle of Superposition is valid, using analogous arguments, for linear differential equations in *more than one variable*, such as the wave equations we shall be considering shortly. In that case, it gives insight into how waves can pass through each other and emerge unchanged.

A Lightly Damped Oscillator

We can go through exactly the same mathematical steps in solving the equation of motion as we did for the heavily damped case: we look for solutions of the form

$$x = x_0 e^{-\alpha t}$$

and as before we find there are solutions with

$$\alpha_1, \alpha_2 = \frac{b \pm \sqrt{b^2 - 4mk}}{2m}.$$

But the difference is that for *light* damping, by which we mean $b^2 < 4mk$, the expression inside the square root is negative! We are going to have to work with the square root of a negative number. We do this formally by writing:

$$\sqrt{b^2 - 4mk} = i\sqrt{4mk - b^2}$$

with $i^2 = -1$ as usual. This gives the two possible exponential solutions:

$$x_1(t) = e^{-\frac{bt}{2m}} e^{-\frac{i\sqrt{4mk-b^2}}{2m}t}, \quad x_2(t) = e^{-\frac{bt}{2m}} e^{+\frac{i\sqrt{4mk-b^2}}{2m}t}.$$

and a general solution

$$x(t) = A_1 e^{-\frac{bt}{2m}} e^{-\frac{i\sqrt{4mk-b^2}}{2m}t} + A_2 e^{-\frac{bt}{2m}} e^{+\frac{i\sqrt{4mk-b^2}}{2m}t}.$$

Of course, the position of the mass $x(t)$ has to be a real number! We must choose A_1 and A_2 to make sure this is so. If we choose

$$A_1 = \frac{1}{2} A e^{-i\delta}, \quad A_2 = \frac{1}{2} A e^{+i\delta}$$

where A and δ are real, and remembering

$$\cos \theta = \frac{1}{2}(e^{+i\theta} + e^{-i\theta}),$$

we find

$$x(t) = A e^{-\frac{bt}{2m}} \cos\left(\frac{\sqrt{4mk-b^2}}{2m}t + \delta\right).$$

This is the most general real solution of the lightly damped oscillator—the two arbitrary constants are the amplitude A and the phase δ . So for small b , we get a cosine oscillation multiplied by a gradually decreasing function, $e^{-bt/2m}$.

This is often written in terms of a **decay time** τ defined by

$$\tau = m/b.$$

The amplitude of oscillation A therefore decays in time as $e^{-t/2\tau}$, and the *energy* of the oscillator (proportional to A^2) decays as $e^{-t/\tau}$. This means that in time τ the energy is down by a factor $1/e$, with $e = 2.71828\dots$

The solution is sometimes written

$$x(t) = Ae^{-\frac{bt}{2m}} \cos(\omega't + \delta)$$

where

$$\omega'^2 = \frac{4mk - b^2}{4m^2} = \frac{k}{m} - \frac{b^2}{4m^2} = \omega_0^2 - \frac{b^2}{4m^2}.$$

Notice that for small damping, the oscillation frequency doesn't change much from the undamped value: the change is proportional to the *square* of the damping.

The Q Factor

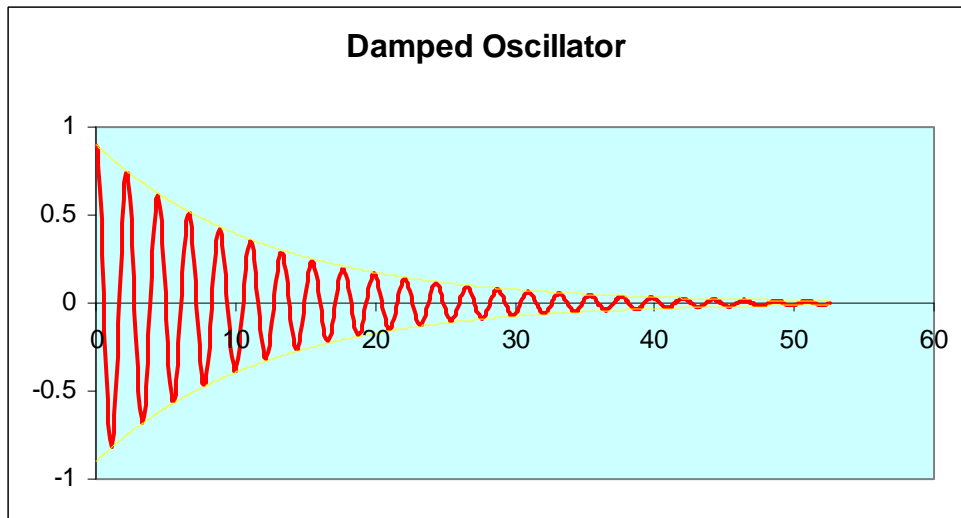
The Q factor is a measure of the “quality” of an oscillator (such as a bell): how long will it keep ringing once you hit it? Essentially, it is a measure of how many oscillations take place during the time the energy decays by the factor of $1/e$.

Q is defined by:

$$Q = \omega_0 \tau$$

so, strictly speaking, it measures how many radians the oscillator goes around in time τ . For a typical bell, τ would be a few seconds, if the note is middle C, 256 Hz, that's $\omega_0 = 2\pi \times 256$, so Q would be of order a few thousand.

Exercise: estimate Q for the following oscillator (and don't forget the energy is proportional to the *square* of the amplitude):



The yellow curves in the graph above are the pair of functions $+e^{-bt/2m}$, $-e^{-bt/2m}$, often referred to as the *envelope* of the oscillation curve, as they “envelope” it from above and below.

*Critical Damping

There is just one case we haven't really discussed, and it's called "critical damping": what happens when $b^2 - 4mk$ is exactly zero? At first glance, that sounds easy to answer: there's just the one solution

$$x(t) = Ae^{-\frac{bt}{2m}}.$$

But that's not good enough—it tells us that if we begin at $t = 0$ with the mass at x_0 , it must have velocity dx/dt equal to $-x_0b/2m$. But, in fact, we can put the mass at x_0 and kick it to any initial velocity we want! So what happened to the other solution?

We can get a clue by examining the two exponentially falling solutions for the overdamped case as we approach critical damping:

$$x(t) = A_1 e^{-\frac{b+b\sqrt{1-\frac{4mk}{b^2}}}{2m}t} + A_2 e^{-\frac{b-b\sqrt{1-\frac{4mk}{b^2}}}{2m}t}$$

As we approach critical damping, the small quantity

$$\varepsilon = \frac{\sqrt{b^2 - 4mk}}{2m}$$

approaches zero. The general solution to the equation has the form

$$x(t) = e^{-\frac{bt}{2m}} (A_1 e^{-\varepsilon t} + A_2 e^{+\varepsilon t}).$$

This is a valid solution for any real A_1, A_2 . To find the solution we're missing, the trick is to take $A_2 = -A_1$. In the limit of small ε , we can take $e^{\varepsilon t} = 1 + \varepsilon t$, and we discover the solution

$$x(t) = -e^{-\frac{bt}{2m}} 2\varepsilon t.$$

As usual, we can always multiply a solution of a linear differential equation by a constant and still have a solution, so we write our new solution as

$$x(t) = A_2 t e^{-\frac{bt}{2m}}.$$

The general solution to the critically damped oscillator then has the form:

$$x(t) = (A_1 + A_2 t) e^{-\frac{bt}{2m}}.$$

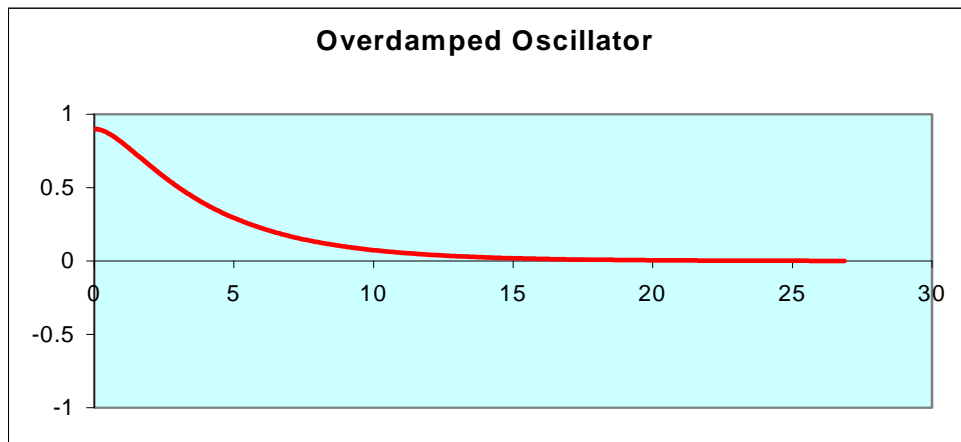
Exercise: check that this is a solution for the critical damping case, and verify that solutions of

the form t times an exponential *don't* work for the other (*noncritical* damping) cases.

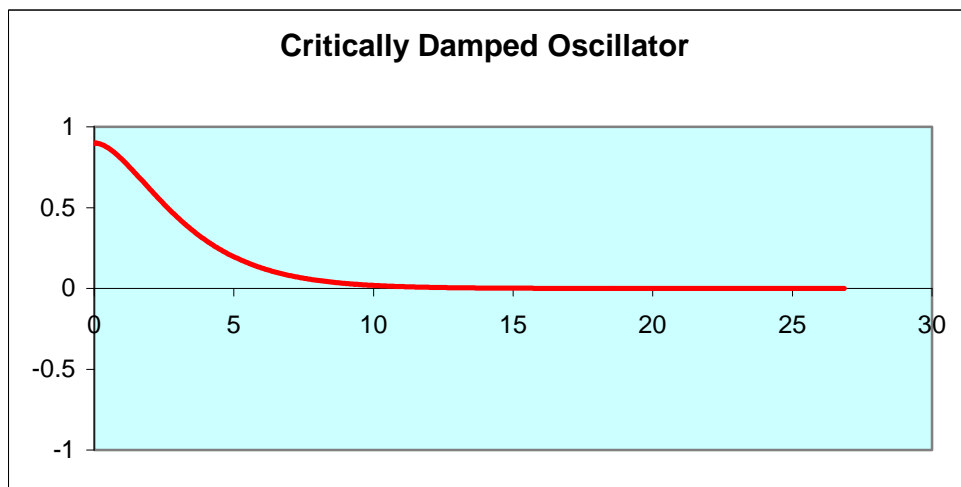
Shock Absorbers and Critical Damping

A shock absorber is basically a damped spring oscillator, the damping is from a piston moving in a cylinder filled with oil. Obviously, if the oil is very thin, there won't be much damping, a pothole will cause your car to bounce up and down a few times, and shake you up. On the other hand, if the oil is *really* thick, or the piston too tight, the shock absorber will be too stiff—it won't absorb the shock, and you will! So we need to tune the damping so that the car responds smoothly to a bump in the road, but doesn't continue to bounce after the bump.

Clearly, the “Damped Oscillator” graph in the Q -factor section above corresponds to too little damping for comfort from a shock absorber point of view, such an oscillator is said to be **underdamped**. The opposite case, **overdamping**, looks like this:

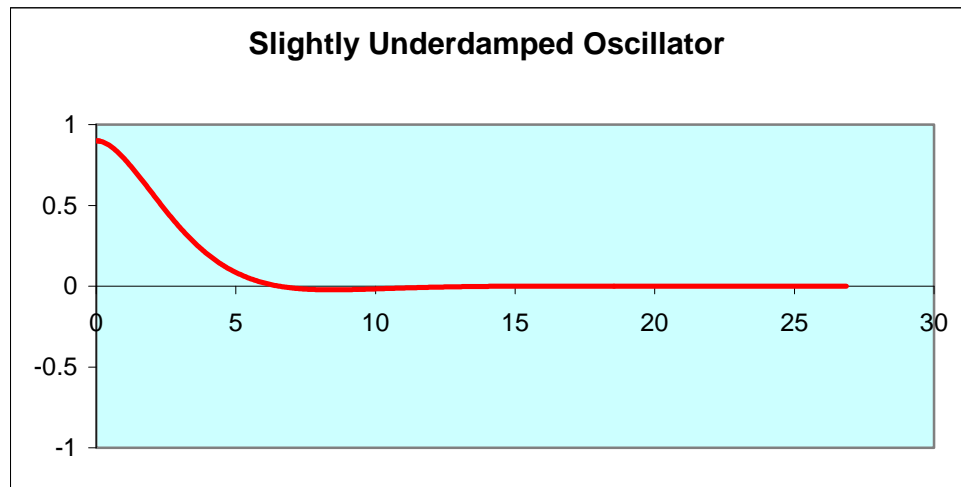


The dividing line between overdamping and underdamping is called **critical damping**. Keeping everything constant except the damping force from the graph above, critical damping looks like:



This corresponds to $\omega' = 0$ in the equation for $x(t)$ above, so it is a purely exponential curve. Notice that the oscillator moves more quickly to zero than in the overdamped (stiff oil) case.

You might think that critical damping is the best solution for a shock absorber, but actually a little less damping might give a better ride: there would be a slight amount of bouncing, but a quicker response, like this:



You can find out how your shock absorbers behave by pressing down one corner of the car and then letting go. If the car clearly bounces around, the damping is too little, and you need new shocks.

A Driven Damped Oscillator: the Equation of Motion

We are now ready to examine a very important case: the driven damped oscillator. By this, we mean a damped oscillator as analyzed above, but with a periodic external force driving it. If the driving force has the same period as the oscillator, the amplitude can increase, perhaps to disastrous proportions, as in the famous case of the [Tacoma Narrows Bridge](#).

The equation of motion for the driven damped oscillator is:

$$m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = F_0 \cos \omega t.$$

We shall be using ω for the frequency of the driving force, and ω_0 for the natural frequency of the oscillator if the damping term is ignored, $\omega_0 = \sqrt{k/m}$.

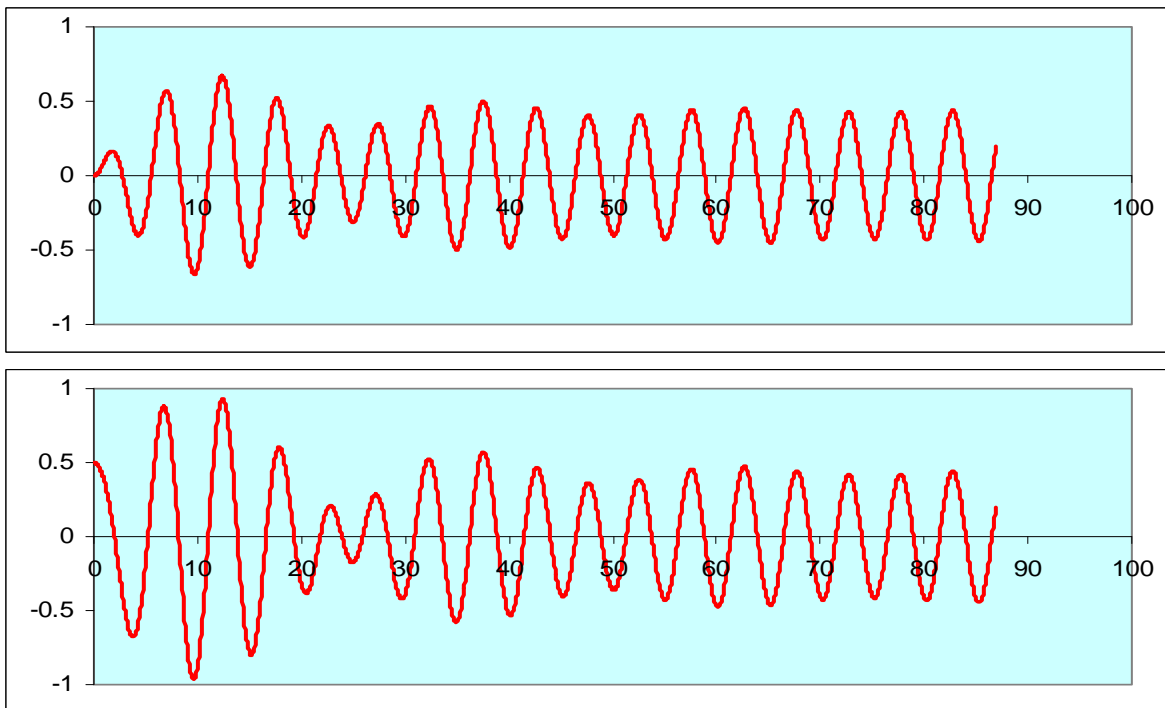
The Steady State Solution and Initial Transient Behavior

The solution to this differential equation is not unique: as with any second order differential equation, there are two constants of integration, which are determined by specifying the initial position and velocity.

However, as we shall prove below using complex numbers, the equation *does* have a *unique* steady state solution with x oscillating at the same frequency as the external drive. How can that be fitted to arbitrary initial conditions? The key is that we can add to the steady state solution

any solution of the *undriven* equation $m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = 0$, and we'll clearly still have a solution of the full damped driven equation. We know what those undriven solutions look like: they all die away as time goes on. So, we can add such a solution to fit the specified initial conditions, and after a while the system will lose memory of those conditions and settle into the steady driven solution. The initial deviations from the steady solution needed to satisfy initial conditions are termed *transients*.

Here's a pair of examples: the same driven damped oscillator, started with zero velocity, once from the origin and once from 0.5:



Notice that after about 70 seconds, the two curves are the same, both in amplitude and phase.

Using Complex Numbers to Solve the Steady State Equation Easily

We begin by writing:

$$\text{external driving force} = F_0 e^{i\omega t}$$

with F_0 real, so the *real* driving force is just the real part of this, $F_0 \cos \omega t$.

So now we're trying to solve the equation

$$m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = F_0 e^{i\omega t}.$$

We'll try the complex function, $x(t) = Ae^{i(\omega t + \varphi)}$, with A a real number, $x(t)$ cycling at the same rate as the driving force. We can always take the amplitude A to be real: that is *not* a restriction, since we've added the adjustable phase factor $e^{i\varphi}$. Physically, this factor allows the solution to lag the driver in phase, as indeed we shall find to be the case. If we succeed in finding an $x(t)$ that satisfies the equation, the real parts of the two sides of the equation must be equal:

If $x(t) = Ae^{i(\omega t + \varphi)}$ is a solution to the equation with the complex driving force, $F_0 e^{i\omega t}$, its real part, $A \cos(\omega t + \varphi)$, will be a solution to the equation with the real driving force, $F_0 \cos \omega t$.

It's very easy to check that $x(t) = Ae^{i(\omega t + \varphi)}$ is a solution to the equation, with the right A and φ ! Just plug it in and see what happens. The differentiations are simple, giving

$$-m\omega^2 Ae^{i(\omega t + \varphi)} + ib\omega Ae^{i(\omega t + \varphi)} + kAe^{i(\omega t + \varphi)} = F_0 e^{i\omega t}.$$

To nail down A and φ , we begin by cancelling out the common factor $e^{i\omega t}$, then shifting the $e^{i\varphi}$ to the other side, to find

$$A = \frac{F_0 e^{-i\varphi}}{k - m\omega^2 + ib\omega}$$

Now A is a real number, and the right hand side of this equation looks alarmingly complex, so what's going on?

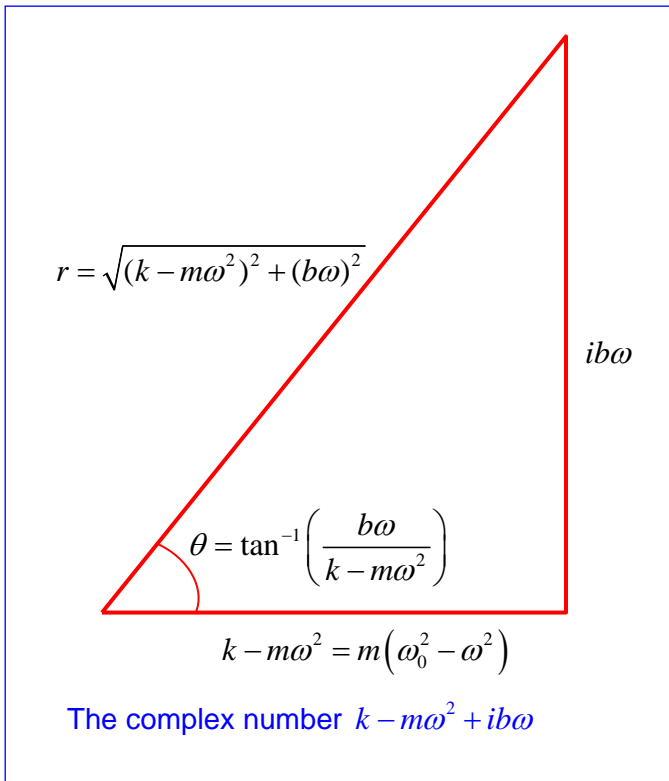
Let's begin to untangle this by diagramming that complex number in the denominator,

$$k - m\omega^2 + ib\omega.$$

It has real part $k - m\omega^2$ and imaginary part $ib\omega$.

Its phase is the angle θ : that is,

$$k - m\omega^2 + ib\omega = re^{i\theta}.$$



Putting it in the equation in this r, θ notation gives

$$A = \frac{F_0 e^{-i\varphi}}{k - m\omega^2 + ib\omega} = \frac{F_0 e^{-i\varphi}}{re^{i\theta}} = \frac{F_0}{r} e^{-i(\varphi + \theta)}.$$

Now, remembering that F_0 and r are real, we see that A will be real (as it must be) if $e^{-i(\varphi+\theta)}$ is real: so $\varphi = -\theta$, and

$$A = \frac{F_0}{r} = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + (b\omega)^2}}, \quad x(t) = Ae^{i(\omega t - \theta)},$$

where we've written $k = m\omega_0^2$.

So we've already solved the differential equation: the amplitude A is proportional to the strength of the driving force, and that ratio is determined by the parameters of the undriven oscillator, and the driving oscillation frequency.

The important thing to note about the amplitude A is that if the damping b is small, A gets *very large* when the frequency of the driver approaches the natural frequency of the oscillator! This is called *resonance*, and is what happened to the Tacoma Narrows Bridge. Of course, it has its positive aspects, from getting a swing going to tuning a radio.

The *phase lag* of the oscillations behind the driver, $\theta = \tan^{-1}(b\omega/(k - m\omega^2))$, is completely determined by the frequency together with the physical constants of the undriven oscillator: the mass, spring constant, and damping strength. So, when the driving force $F_0e^{i\omega t}$ generates the motion $x(t) = Ae^{i(\omega t + \varphi)} = Ae^{i(\omega t - \theta)}$, the lag angle θ is independent of the strength of the driving force: a stronger force doesn't get the oscillator more in sync, it just increases the amplitude of the oscillations.

Note that at *low* frequencies, $\omega \ll \omega_0$, the oscillator lags behind by a small angle, but at resonance $\omega = \omega_0$ $\theta = \pi/2$, and for driving frequencies above ω_0 , $\theta > \pi/2$.

Back to Reality

To summarize: we've just established that $x(t) = Ae^{i(\omega t - \theta)}$ with $A = F_0 / \sqrt{m^2(\omega_0^2 - \omega^2)^2 + (b\omega)^2}$ and $\theta = \tan^{-1}(b\omega/(k - m\omega^2))$ is a solution to the driven damped oscillator equation

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = F_0 e^{i\omega t}$$

with the complex driving force $F_0 e^{i\omega t}$.

So, equating the real parts of the two sides of the equation, since m , b , k are all real,

$$x = A \cos(\omega t - \theta)$$

is a solution of the equation with the real driving force $F_0 \cos \omega t$.

We could have found this out without complex numbers, by using a trial solution $A \cos(\omega t + \varphi)$. However, it's not that easy—the left hand side becomes a mix of sines and cosines, and one needs to use trig identities to sort it all out. With a little practice, the complex method is easier and is certainly more direct.

Now the total energy of the oscillator is

$$\begin{aligned} E &= \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \\ &= \frac{1}{2}mv^2 + \frac{1}{2}m\omega_0^2x^2. \end{aligned}$$

Putting in

$$x(t) = A \cos(\omega t - \theta), \quad v(t) = -A\omega \sin(\omega t - \theta)$$

gives

$$E = \frac{1}{2}mA^2 \left(\omega^2 \sin^2(\omega t - \theta) + \omega_0^2 \cos^2(\omega t - \theta) \right).$$

Note that this is *not* constant through the cycle unless the oscillator is at resonance, $\omega = \omega_0$.

We can see from the above that at the resonant frequency, $E = \frac{1}{2}m\omega_0^2A^2$, and from the section above

$$A = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + (b\omega)^2}},$$

so the energy in the oscillator at the resonant frequency is

$$E_{\text{resonance}} = \frac{1}{2}m\omega_0^2A^2 = \frac{1}{2}m\omega_0^2 \frac{F_0^2}{b^2\omega_0^2} = \frac{1}{2}m \frac{F_0^2}{b^2} = \frac{Q^2}{2} \frac{F_0^2}{m\omega_0^2},$$

recalling that $Q = \omega_0\tau = \omega_0m/b$.

So Q , the quality factor, the measure of how long an oscillator keeps ringing, also measures the strength of response of the oscillator to an external driver at the resonant frequency.

But what happens on going *away* from the resonant frequency? Let's assume that Q is large, and the driving force is kept constant. It won't take much change in ω from ω_0 for the denominator $m^2(\omega_0^2 - \omega^2)^2 + (b\omega)^2$ in the expression for E to double in size. In fact, for large Q , it's a good approximation to replace $b\omega$ by $b\omega_0$ over that variation, and it is then straightforward to check that the energy in the oscillator drops to one-half its resonant value for $\omega - \omega_0 \cong \pm\omega_0/2Q$.

Exercise: prove this.

The bottom line is that for increasing Q , the response at the resonant frequency gets larger, but this large response takes place over a narrower and narrower range in driving frequencies.

And Now to Work...

An important practical question is: how much *work* is the driver doing to keep this thing going?

It's simplest to work with the real solution. Suppose the oscillator moves through Δx in a time Δt , the driving force does work $(F_0 \cos \omega t) \Delta x$, so

$$\text{rate of working at time } t = (F_0 \cos \omega t)(\Delta x / \Delta t) = (F_0 \cos \omega t)v(t)$$

The important thing is the *average* rate of working of the driving force, the *mean power input*, found by averaging over a complete cycle:

From $x(t) = A \cos(\omega t - \theta)$, $v(t) = -A\omega \sin(\omega t - \theta)$, averaging the power input (the bar above means average over a complete cycle) and denoting average power by P ,

$$\begin{aligned} P &= \overline{F_0(\cos \omega t)v(t)} \\ &= -F_0 A \omega \overline{\cos \omega t \sin(\omega t - \theta)} \\ &= -F_0 A \omega \overline{\cos \omega t \sin \omega t \cos \theta} + F_0 A \omega \overline{\cos^2 \omega t \sin \theta} \\ &= \frac{1}{2} F_0 A \omega \sin \theta \end{aligned}$$

since over one cycle the average $\overline{\cos^2 \omega t} = \frac{1}{2}$ and $\overline{\cos \omega t \sin \omega t} = \frac{1}{2} \overline{\sin 2\omega t} = 0$ (Remembering $\cos^2 \omega t + \sin^2 \omega t = 1$ at all times, and sine is just cosine moved over, so they must have the same average over a complete cycle.)

This can be expressed entirely in terms of the driving force and frequency. Since

$$A = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + (b\omega)^2}}, \quad \sin \theta = \frac{b\omega}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + (b\omega)^2}},$$

$$\begin{aligned} P &= \frac{1}{2} F_0 A \omega \sin \theta \\ &= \frac{1}{2} \frac{b\omega^2 F_0^2}{m^2(\omega_0^2 - \omega^2)^2 + (b\omega)^2} \end{aligned}$$

Exercise 1: Prove that for a lightly damped oscillator, at resonance the oscillator extracts the most work from the driving force.

Exercise 2: Prove that any solution of the damped oscillator equation (with $F = 0$) can be added to the driven oscillator solution, and gives another solution to the driven oscillator. How do you pick the "right solution"?

[previous](#) [index](#) [next](#)