Oscillations: the Essentials

3/22/07

Complex Numbers

Be familiar with the complex plane: $x+iy=re^{i\theta}$, definitions: mod $z=|z|=r$, phase of $z=\theta$. To multiply complex numbers, multiply the mods, *add* the phases. Know the all-important formula:

$$e^{i\theta}=\cos\theta+i\sin\theta,$$

and be able to interpret it as a point on the *unit circle*.

Some useful small x approximations:

$$e^x\cong 1+x,\quad \ln(1+x)\cong x,\quad \sqrt{1+x}\cong 1+\tfrac{1}{2}x,\quad (1+x)^{-1}\cong 1-x$$

and for small angles

$$\sin\theta\cong\tan\theta\cong\theta,\quad \cos\theta\cong 1-\tfrac{1}{2}\theta^2.$$

Undamped Simple Harmonic Oscillator:

Be able to solve the equation

$$F=ma,\ \text{or}\ m\frac{d^2x}{dt^2}=-kx,$$

and write down the velocity and kinetic energy at any time. Be able to sketch a graph of the *potential energy* as a function of position, both for a horizontal and a vertical spring. Be able to derive the angular frequency and the period. Know how to find the dependence of the period on *k*, *m* using dimensional arguments.

A Heavily Damped Oscillator:

You should know the equation of motion

$$m\frac{d^2x}{dt^2}=-kx-b\frac{dx}{dt}$$

and that a solution is $x=x_0e^{-\alpha t}$. Be able to substitute this in the equation to find α and from that the condition $b^2>4mk$ for exponential decay.
Be able to use dimensional arguments to find that m/b and b/k *both* have dimensions of time, and for *large* damping give a physical interpretation of when these very different times are physically relevant.

A Lightly Damped Oscillator:

I would not ask you to solve the equation of motion in this case, but you should know the following facts: the energy in the oscillator decays in time as $e^{-t/\tau}$, where $\tau=m/b$. The Q factor $Q=\omega_0\tau$ measures how many radians the oscillator goes through during the time the energy drops by a factor $1/e$. You should be able to sketch how the amplitude decays in time, showing the exponential "envelope" of the oscillations.

Critical Damping:

Know the condition for critical damping, and be able to sketch how the amplitude decays above, at and below critical damping.

Principle of Superposition:

For the damped simple harmonic oscillator reviewed above, if $f_1(x)$ and $f_2(x)$ are solutions, so is $A_1f_1(x) + A_2f_2(x)$ for any constants A_1, A_2. However, this is *not* the case for the driven oscillator reviewed below: in fact, if $f_1(x)$ is a solution for the driven oscillator, $2f_1(x)$ would be a solution if the driving force were doubled. But one *can* add to a solution of the driven oscillator any solution of the same but undriven oscillator—and this is usually necessary to fit initial conditions.

A Driven Damped Oscillator:

Be able to write down the equation of motion $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \cos \omega t$ and understand it as the real part of the equation $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 e^{i\omega t}$. Know how the put in the trial solution $x(t) = Ae^{i(\omega t + \varphi)}$ and from that deduce the amplitude $A = \frac{F_0}{r} = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + (b\omega)^2}}$. Be familiar with the form of this as a function of driving frequency, especially near resonance.

The Pendulum:

Know the equation of motion of a simple pendulum, how it relates to a simple harmonic oscillator, how to handle the equation if the pendulum is a rigid body rather than just a point mass on a light rod or string. Know how to find the potential energy as a function of angle.