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## Stokes' Law and the Coffee Filters

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### A Problem

We found that Stokes' Law, which we derived in the form

$$F_{\text{drag}} = C\eta v$$

from purely dimensional considerations (Stokes did the hard part of proving that  $C = 6\pi$ ) correctly predicted that for two small steel balls, one having a radius exactly twice the other, the bigger one would fall through a fluid four times faster (it had eight times the weight, and twice the drag force for the same velocity, and the drag force is proportional to the velocity).

Now let us ask what Stokes' Law predicts for the following *coffee filter experiment*:

If we drop a single coffee filter, it reaches a terminal velocity of about 0.8 meters per sec after falling less than a meter. If we drop a stack of four close packed filters, the terminal velocity clocks in at about 1.6 meters per sec.



That is to say, the stack of *four* filters has a terminal velocity *twice* that of a single filter. Now at terminal velocity the drag force is exactly balancing the weight of the object falling. The stack of four filters is almost indistinguishable in shape and size from the single filter, so it's difficult to believe there's any significant difference in the air flow pattern round the falling filters for the same speed. Therefore the Stokes' drag from the air friction should be the same  $C\eta v$  for both.

(We can't say  $C = 6\pi$ , that was derived for a falling *sphere*, but the *dimensional* argument should still be working.) Yet this implies that the terminal velocity of the stack of four filters should be *four times* the terminal velocity of the single filter!

What is wrong with our dimensional analysis? It worked brilliantly for the little steel balls, but seems to have flunked the coffee filter test. In what respect are these two experiments different?

### Another Kind of Drag Force

Perhaps the best way to see what is wrong is to do the steel ball experiment on a completely different scale. Let us imagine dropping a cannonball from an airplane. This will also reach a terminal velocity, but at hundreds of miles an hour. However, in contrast to the steel balls in glycerin experiment, it turns out that this time the viscous drag is not the important effect. *At high speeds, most of the work done by the falling body is in just pushing the air out of the way.*

Let us estimate how much force the cannonball exerts on the air pushing it out of its path. Suppose the cannonball is falling at steady speed  $v$ , and it has radius  $a$ . Then it has to move aside a volume  $\pi a^2 v$  of air per second, and this air will be moved at a speed of order of magnitude  $v$ . Therefore, the rate at which the cannonball imparts momentum to the air (which was previously at rest) is of order  $\pi \rho a^2 v^2$  per second. But the rate of change of momentum per second is just the force, so the cannonball is pushing the air with a force of order  $\pi \rho a^2 v^2$ . By Newton's Third Law, Action = Reaction, this is also the drag force the cannonball experiences as it falls at  $v$ .

*Exercise:* Assuming the drag force depends only on  $v$ ,  $a$ , and the density of air  $\rho$ , use a dimensional argument to show it must have this form.

### So What is the Real Drag Force?

Using purely dimensional considerations, we have derived two quite different formulas for the drag force on a sphere falling through a fluid:

Viscous drag force:

$$F_{\text{viscous}} = C a \eta v$$

and inertial drag force:

$$F_{\text{inertial}} = C' \rho a^2 v^2.$$

We call the second "inertial" because it arises from just pushing the still air out of the way, and would be the same if the air had no viscosity at all.

The truth is that the two different derivations we have presented above for these two different drag forces are both too simple. In fact, in real situations, *both* types of forces are present. This does not mean, though, that we can simply add the forces with suitable coefficients—the general situation is far more complicated. However, it can be described mathematically by a complicated differential equation, the Navier-Stokes equation. The good news is that the solutions to this equation for a given flow configuration, such as flow past a sphere, or flow past a wing, can be classified in terms of a single dimensionless parameter, the Reynolds number.

**The Reynolds number is just the ratio of the inertial drag to the viscous drag:**

$$N_R = 2a\rho v/\eta.$$

The factor of 2 is the standard definition of the Reynolds number—this is just a matter of convention, it is of course not fixed by the dimensional arguments. And the Reynolds number is **dimensionless**: it's the ratio of two forces, so will be the same in any system of units!

The theoretical prediction from the Navier-Stokes equation that the flow pattern in a given geometry depends only on the Reynolds number is well established experimentally, and makes it possible to find how air flows around an airplane in flight by testing a scale model in a wind tunnel, adjusting wind speed to get the same Reynolds number.

Stokes' Law for a falling sphere is found experimentally to be reasonably accurate for  $N_R$  less than or of order 1.

*Reference:* The derivation of Stokes' Law (the  $6\pi$ ) can be found, for example, in G. K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge, 1967, 2000.

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