Physics 751 Final

18 December 2007

Only hand in answers to *five* questions!

Possibly useful info:

Simple Harmonic Oscillator:

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + i\frac{p}{m\omega}\right), \quad a^\dagger \,|\, n\rangle = \sqrt{n+1}\,|\, n+1\rangle$$

Angular Momentum Operators:

$$[J_i, J_j] = i\hbar\varepsilon_{ijk}J_k, \quad J^2 \,|\, jm\rangle = \hbar^2 j(j+1)\,|\, jm\rangle \quad J_z \,|\, jm\rangle = \hbar m \,|\, jm\rangle$$

$$J_\pm = J_x \pm iJ_y, \quad J_\pm \,|\, jm\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\,|\, j, m\pm 1\rangle$$

$$Y_0^0 = (4\pi)^{-1/2}, \quad Y_1^{\pm 1} = \mp(3/8\pi)^{1/2}\sin\theta e^{\pm i\varphi}, \quad Y_1^0 = (3/4\pi)^{1/2}\cos\theta$$

$$Y_2^{\pm 2} = (15/32\pi)^{1/2}\sin^2\theta e^{\pm 2i\varphi}, \quad Y_2^{\pm 1} = \mp(15/8\pi)^{1/2}\sin\theta\cos\theta e^{\pm i\varphi}, \quad Y_2^0 = (5/16\pi)^{1/2}\left(3\cos^2\theta - 1\right)$$

$$\vec{\sigma} = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

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1. A particle of mass m in one dimension is in the potential

$$V(x) = \lambda_1\delta(x) + \mu_1\left(\delta(x+a) + \delta(x-a)\right).$$

(a) Taking $\lambda = 2m\lambda_1/\hbar^2$, $\mu = 2m\mu_1/\hbar^2$, find the boundary conditions satisfied by the wave function at 0, a for states of negative or zero energy.

(b) Assuming all the delta functions are attractive potentials, sketch *qualitatively* the ground state wave function when all the potentials are very weak but of comparable strengths.

(c) Find the ranges of values of λ and μ for which a *second* bound state exists. (Include the possibility of repulsive as well as attractive potentials.)

(d) Are there any values of λ and μ for which the ground state and the second bound state are extremely close in energy? If you say yes, give an example.

(e) Give a qualitative sketch of a possible third bound state, for appropriate values of the potentials. Is it possible to have a third bound state if the central potential is repulsive?

2. (a) For $a^\dagger$, a the usual simple harmonic oscillator creation and annihilation operators, and $f(x)=\sum c_n x^n$, prove that $\left[a, f\left(a^\dagger\right)\right]=f'\left(a^\dagger\right)$.

(b) Use your result in (a) to prove that the eigenstate of the position operator can be written

$$|x\rangle = A\exp\left[\left(-\frac{1}{2}\right)\left(a^\dagger - x\sqrt{\frac{2m\omega}{\hbar}}\right)^2\right]|0\rangle$$

where A is a normalization constant, and $|0\rangle$ means the ground state. (NOTE: the x in the exponential here is just a number, *not* an operator.)

(c) For the position eigenket located at the origin, which we write $|x=0\rangle$ to minimize confusion, sketch the first few contributions from the SHO eigenstates and indicate how they build towards a delta function.

3. A path integral calculation of the propagator for the simple harmonic oscillator gives

$$U(x,T;x',0)=\sqrt{\frac{m\omega}{2\pi\hbar i\sin\omega T}}\exp\left\{\frac{im\omega}{2\hbar\sin\omega T}\left[\left(x^2+x'^2\right)\cos\omega T-2xx'\right]\right\}.$$

The standard expression for the oscillator in terms of the eigenstates is

$$U(x,T;x',0)=\sum_{n=1}^{\infty}e^{-iE_nT/\hbar}\langle x|\psi_n\rangle\langle\psi_n|x'\rangle.$$

(a) Beginning with the path integral expression for the oscillator, setting $x=x'=0$, and making a suitable expansion, prove that the eigenenergies contributing to the propagator are $E=\hbar\omega/2,\ 3\hbar\omega/2,\ 5\hbar\omega/2,\dots$. What happened to the levels in between?

(b) Now consider the extraction of the eigenfunctions. Let $x=x'$. Find $E_0, E_1, |\psi_0(x)|^2, |\psi_1(x)|^2$ by expanding in powers of $\alpha=\exp(i\omega T)$.

4. A particle has spin angular momentum $\frac{3}{2}\hbar$ and orbital angular momentum $\hbar$.

(a) What are the possible allowed values of the total angular momentum?

(b) What is the dimensionality of the total angular momentum space?

(c) Explain how the rotation operator in the product space of the two angular momenta can be written as a sum of irreducible representations.

(d) Find explicit expressions for the normalized eigenstates $\left| j,m \right\rangle$ of the total angular momentum operators $\vec{J}^2, J_z$ for $m > 0$, in terms of product kets $\left|\frac{3}{2},m\right\rangle \otimes \left|1,m'\right\rangle$.

5. A spin ½ is in the up eigenstate in the direction $\frac{1}{\sqrt{3}}(1,1,1)$.

(a) What is the probability of measuring the spin as up in the x-direction?

(b) What is the probability of measuring the spin as up in the z-direction?

6. An electron is confined to move in a plane, with a perpendicular uniform magnetic field having vector potential $(0, Bx, 0)$.

(a) Write down the appropriate Schrödinger equation.

(b) Argue that the eigenfunctions can be factorized as $f(x)g(y)$, giving your justification, and solve for the simpler of the two functions f, g assuming periodic boundary conditions on a rectangle $L_x \times L_y$.

(c) Without working out the solution to the equation for the other function explicitly, explain how you can find the degeneracy of the lowest energy level—the total number of states in that level—within the rectangle.