

Adding Angular Momenta

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Introduction

Consider a system having two angular momenta, for example an electron in a hydrogen atom having both orbital angular momentum and spin. The ket space for a single angular momentum has an orthonormal basis $|j, m\rangle$ so for two angular momenta an obvious orthonormal basis is the set of direct product kets $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$. What does this mean, exactly? Suppose the first angular momentum \vec{J}_1 has magnitude $\vec{J}_1^2 = \hbar^2 j_1(j_1 + 1)$, and is in the state $\sum_{m_1=-j_1}^{j_1} \alpha_{m_1} |j_1, m_1\rangle$, and similarly the second angular momentum \vec{J}_2 is in the state $\sum_{m_2=-j_2}^{j_2} \beta_{m_2} |j_2, m_2\rangle$. Evidently the probability amplitude for finding the first spin in state m_1 and at the same time the second in m_2 is $\alpha_{m_1} \beta_{m_2}$, and we denote that state by $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$. How to handle these direct product spaces will become clear on examining specific examples, as we do below, beginning with two spins one-half.

Now the sum of two angular momenta

$$\vec{J} = \vec{J}_1 + \vec{J}_2$$

is *itself* an angular momentum, operating in a space with a complete basis $|j, m\rangle$.

This is easy to prove: the components of \vec{J}_1 satisfy $[J_{1i}, J_{1j}] = i\hbar \varepsilon_{ijk} J_{1k}$, and similarly for the components of \vec{J}_2 . The components of \vec{J}_1 commute with the components of \vec{J}_2 , of course, from which it follows immediately that the vector components of $\vec{J} = \vec{J}_1 + \vec{J}_2$ do indeed obey the angular momentum commutation relations: and recall that the commutation relations were sufficient to determine the allowed sets of eigenvalues.

We shall prove later that the eigenstates $|j, m\rangle$ of \vec{J}^2, \vec{J}_z are a complete basis for the product space of the eigenkets of $\vec{J}_1^2, \vec{J}_2^2, J_{1z}, J_{2z}$ —to establish this, we must first find the possible allowed values of the total angular momentum quantum number j .

Here we have, then, two different orthonormal bases for what is evidently the same vector space. In practical applications, it often turns out that we have to translate from one of these bases to the other. Our present task is to construct the appropriate transformation: we accomplish this by finding the coefficients of any $|j, m\rangle$ in the $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$ basis. (These are called the *Clebsch-Gordan* coefficients.)

We shall build gradually, beginning with adding two spins one-half, then a spin one-half with an orbital angular momentum, finally two general angular momenta. This is a very important part of quantum mechanics: we give every detail. Readers already somewhat familiar with the subject may find this a bit tedious, they can glance over the introductory examples and go to the general case.

Adding Two Spins: the Basis States and Spin Operators

The most elementary example of a system having two angular momenta is the hydrogen atom in its ground state. The orbital angular momentum is zero, the electron has spin angular momentum $\frac{1}{2}\hbar$, and the proton has spin $\frac{1}{2}\hbar$.

The space of possible states of the electron spin has the two basis kets $|\uparrow\rangle_e$ and $|\downarrow\rangle_e$, (also variously written as $|+\rangle, |-\rangle; \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \chi_+, \chi_-!$) the basis proton spin kets are $|\uparrow\rangle_p$ and $|\downarrow\rangle_p$, so the possible states of the combined system are kets in the direct product space which has a basis of four kets:

$$|\uparrow_e \uparrow_p\rangle, |\uparrow_e \downarrow_p\rangle, |\downarrow_e \uparrow_p\rangle, |\downarrow_e \downarrow_p\rangle$$

using $|\uparrow_e \uparrow_p\rangle$ as shorthand for $|\uparrow\rangle_e \otimes |\uparrow\rangle_p$.

Note here that we've written the kets in "alphabetical order" with \uparrow as the first letter, \downarrow as the second. That is to say, we've first written all the kets having \uparrow as the first letter, etc.

For the more general case of adding j_1 to j_2 , to be considered shortly, we'll order the kets in the same "alphabetical" way, writing first all the kets having $m_1 = j_1$, and so on down to $m_1 = -j_1$, so the possible sets $m_1 m_2$ are:

$$(j_1 j_2, j_1(j_2 - 1), \dots, j_1(-j_2), (j_1 - 1)j_2, (j_1 - 1)(j_2 - 1), \dots, (j_1 - 1)(-j_2), \dots, (-j_1)(-j_2)).$$

The dimensionality of this space is then $(2j_1 + 1) \times (2j_2 + 1)$.

Now the first block of $2j_2 + 1$ elements all have the same m -component of j_1 , that is, $m_1 = j_1$, the next block has $m_1 = j_1 - 1$, and so on. Think about what this means for constructing a *rotation operator* acting on the kets in this space: if it operates only on the angular momentum j_1 , it will change the factors m_i multiplying the blocks, if the operator rotates only j_2 , it will operate *within* each block, all the blocks being changed in the same way.

To get a feeling for how this works in practice, we go back to the simplest case, two spins one-half.

The space is four-dimensional, having basis

$$|\uparrow_e \uparrow_p\rangle, |\uparrow_e \downarrow_p\rangle, |\downarrow_e \uparrow_p\rangle, |\downarrow_e \downarrow_p\rangle.$$

Any operator acting on the spins will be represented by a 4×4 matrix, best thought of as a 2×2 matrix made up of 2×2 blocks: an operator acting on the proton spin acts *within* the blocks, one operating on the electron spin acts on the blocks themselves, regarded as single entities.

Let's look at a few examples. Recall that the raising operator for a single spin is the 2×2 matrix $S^+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. So what is the raising operator for the *electron* spin?

$$S_e^+ \otimes I_p = \hbar \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \hbar \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We use bold to denote 2×2 matrices.

The pattern is clear: the big structure (in **bold** above), that of the four 2×2 blocks, reflect the structure of the electron spin operator $S^+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, within those blocks (of which only one survives) the identity operator $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ acts on the proton spin.

The operator that raises the *proton* spin is:

$$I_e \otimes S_p^+ = \hbar \begin{pmatrix} \boldsymbol{\sigma}^+ & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma}^+ \end{pmatrix} = \hbar \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

What about the operator that raises *both* electron and proton spin? In this case, the pattern of blocks, and the pattern within each block, must both be $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, so

$$S_e^+ \otimes S_p^+ = \hbar^2 \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma}^+ \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \hbar^2 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

There is only one nonzero matrix element because only one member of the base survives this operation.

If two spins interact (via their magnetic moments, for example) in a way that preserves total angular momentum, a possible term in the Hamiltonian would be $S_e^- S_p^+$, represented by:

$$S_e^- S_p^+ = \hbar \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \hbar S_e^- \otimes S_p^+.$$

Representing the Rotation Operator for Two Spins

Recall from the lecture on spin that the rotation operator on a single spin one-half is

$$D^{(1/2)}\left(R(\theta \hat{n})\right) = e^{-\frac{i\theta \hat{n} \cdot \vec{J}}{\hbar}} = e^{-i(\theta/2)(\hat{n} \cdot \vec{\sigma})} = I \cos \frac{\theta}{2} - i(\hat{n} \cdot \vec{\sigma}) \sin \frac{\theta}{2}$$

in the 2×2 spinor space. As we established, this matrix operator has the form

$$\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) - in_z \sin(\theta/2) & -(in_x + n_y) \sin(\theta/2) \\ -(in_x - n_y) \sin(\theta/2) & \cos(\theta/2) + in_z \sin(\theta/2) \end{pmatrix}.$$

with $|a|^2 + |b|^2 = 1$.

This set of unitary 2×2 matrices form a *representation of the rotation group* in the sense that the total resulting from two successive rotations is given by the matrix which is the matrix product of those corresponding to the two rotations.

From the discussion in the previous section, it should be clear that in the product space of the two spins, the representation of the rotation operator—both spins of course undergoing the same rotation—is:

$$\begin{pmatrix} a \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} & b \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \\ -b^* \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} & a^* \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \end{pmatrix} = \begin{pmatrix} a^2 & ab & ab & b^2 \\ -ab^* & aa^* & -bb^* & a^*b \\ -ab^* & -bb^* & aa^* & a^*b \\ b^2 & -a^*b^* & -a^*b^* & a^{*2} \end{pmatrix}.$$

This set of 4×4 matrices, again with $|a|^2 + |b|^2 = 1$, must also form a *representation of the rotation group* over the four-dimensional space. We shall shortly discover that this representation can be simplified, but to achieve that we need to analyze the states in terms of *total* angular momentum.

Representing States of Two Spins in Terms of Total Angular Momentum

We're now ready to look at total spin states for the ground-state (zero *orbital* angular momentum) hydrogen atom.

Consider first the state with both electron and proton spin pointing upwards, $|\uparrow\uparrow\rangle$. The z -component of the total spin is $S_z = S_z^e + S_z^p$, so $S_z = \hbar$. Labeling the total spin state $|s, m\rangle$, we have a state with $m = 1$, so $s = 1$. (To confirm that this state indeed has $s = 1$ we can apply the total-spin raising operator $S_+ = S_+^e + S_+^p$. Since both component spins have maximum m value, $S_+ |s, 1\rangle_{sm} = (S_+^e + S_+^p) |\uparrow\uparrow\rangle = 0$, but S_+ only gives zero when acting on the $m = s$ member of a multiplet.)

We find, then, that $|\uparrow\uparrow\rangle = |1, 1\rangle_{sm}$ where we've added the suffix sm to make clear that the numbers in the last ket signify $|s, m\rangle$ for the total spin. The total spin $s = 1$, being a total angular momentum eigenstate, has a triplet of m values, $m = 1, 0, -1$, $|1, 1\rangle_{sm}$ being the top member. The $m = 0$ member is found by applying the lowering operator to $|\uparrow\uparrow\rangle$:

$$\begin{aligned} S_- |\uparrow\uparrow\rangle &= (S_-^e + S_-^p) (|\uparrow\rangle_e \otimes |\uparrow\rangle_p) \\ &= \hbar |\downarrow\rangle_e \otimes |\uparrow\rangle_p + \hbar |\uparrow\rangle_e \otimes |\downarrow\rangle_p \end{aligned}$$

which together with

$$S_- |\uparrow\uparrow\rangle = S_- |1, 1\rangle_{sm} = \sqrt{2\hbar} |1, 0\rangle_{sm},$$

gives

$$|1, 0\rangle_{sm} = \frac{1}{\sqrt{2}} (|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle).$$

Obviously, the third member of the triplet, $|1, -1\rangle_{sm} = |\downarrow\downarrow\rangle$.

But this triplet only accounts for *three* basis states in the $|s, m\rangle$ total angular momentum representation. A fourth state, orthogonal to these three and normalized, is $\frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$. This has $m = 0$, and also has $s = 0$, easily checked by noting that the total spin raising operator

$S_+ = S_+^e + S_+^p$ acting on this state gives zero, so the state has the maximum allowed m for its s value.

To summarize: in the total angular momentum $|s, m\rangle$ representation for two spins one-half, the four basis states are $|1, 1\rangle_{sm}$, $|1, 0\rangle_{sm}$, $|1, -1\rangle_{sm}$, $|0, 0\rangle_{sm}$. This orthonormal basis spans the same space as the other orthonormal set $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$. Our construction of the $|s, m\rangle$ states above amounts to finding one set of basis kets in terms of the others.

Note that since both sets of basis kets are orthonormal, mapping a vector from one set to the other is a unitary transformation. But there's more: the coefficients we found expressing one basis ket in the other basis are all *real*. This means that if any ket has real coefficients in one basis, it does in the other. For this special case of all real coefficients, a unitary transformation is termed *orthogonal*.

The orthogonal transformation expressing one base in terms of the other is easy to construct:

$$\begin{pmatrix} |1, 1\rangle_{sm} \\ |1, 0\rangle_{sm} \\ |0, 0\rangle_{sm} \\ |1, -1\rangle_{sm} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} |\uparrow\uparrow\rangle \\ |\uparrow\downarrow\rangle \\ |\downarrow\uparrow\rangle \\ |\downarrow\downarrow\rangle \end{pmatrix}.$$

The matrix is orthogonal and symmetric, so is its own inverse.

Geometrically, $s = 1$ means the component spins are parallel, for $s = 0$ they are antiparallel. This can be stated more precisely: $2\vec{S}_1 \cdot \vec{S}_2 = S^2 - S_1^2 - S_2^2$, so for $s = 1$, $\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\hbar^2 (2 - \frac{3}{4} - \frac{3}{4}) = \hbar^2 / 4$, and for $s = 0$ $\vec{S}_1 \cdot \vec{S}_2 = -\frac{3}{4}\hbar^2$. This makes it easy to construct projection operators into the $s = 0$ and $s = 1$ subspaces: $P_{s=1} = (\vec{S}_1 \cdot \vec{S}_2 / \hbar^2) + \frac{3}{4}$.

Representing the Rotation Operator in the Total Angular Momentum Basis

We've already established that the rotation operator, acting on the two spin system, can be represented by a 4×4 matrix, and that the new (total angular momentum) basis can be reached from the original (two separate spin) basis by the orthogonal transformation given explicitly above. Therefore, pre-and post-multiplying the two-spin rotation operator will in fact give a 4×4 matrix representation of the rotation operator in the new total angular momentum basis.

However, that approach misses the point: first, the singlet state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ has *zero* angular momentum, and so is *not changed by rotation*.

Second, the triplet state has angular momentum one, so rotation operators must act on it just as we found earlier for an angular momentum one:

$$D^{(1)}\left(R(\vec{\theta})\right) = e^{\frac{i\theta\hat{n}\cdot\vec{J}}{\hbar}} = I + (\cos\theta - 1)\left(\frac{\hat{n}\cdot\vec{J}}{\hbar}\right) - i\sin\theta\left(\frac{\hat{n}\cdot\vec{J}}{\hbar}\right).$$

This means that, as far as rotations are concerned, the space spanned by the four kets $|0,0\rangle_{sm}, |1,1\rangle_{sm}, |1,0\rangle_{sm}, |1,-1\rangle_{sm}$ is actually a *sum* of two separate subspaces, the one-dimensional space $|0,0\rangle_{sm}$, and the three-dimensional space having basis $|1,1\rangle_{sm}, |1,0\rangle_{sm}, |1,-1\rangle_{sm}$. Under rotation, a vector in one of these subspaces stays there: there are no cross terms in the matrix mixing the spaces.

This means that the rotation matrix has the form $\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_3 \end{pmatrix}$ where \mathbf{R}_3 is the 3×3 matrix for spin one, \mathbf{I} is just the 1×1 trivial matrix in the singlet subspace, in other words 1, and the $\mathbf{0}$'s are 1×3 and 3×1 sets of zeroes.

A state of the spins can of course be a sum of components in the two subspaces, for example

$$|\uparrow\downarrow\rangle = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\right) + \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle).$$

Reducible and Irreducible Group Representations

We began our discussion of two spins one-half by examining properties of spin operators in the four-dimensional product space of the two two-dimensional spin spaces, and went on to construct a four-dimensional representation of the general rotation operator in that space: a matrix representation of the rotation group. But when the two-spin system is labeled in terms of total angular momentum, we find that in fact this four-dimensional rotation operator is a sum of a three-dimensional rotation, and a trivial identity rotation for an angular momentum zero state. The four-dimensional operator can be “diagonalized”: the space split into a three dimensional space and a one-dimensional space that don’t mix under rotation, and any state of the system is a sum of kets from the two spaces.

This is often expressed by saying the *product* space of two spins one-half is the *sum* of a spin one space and a spin zero space, and written

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0.$$

Putting in the dimensionalities of the spaces in this equation,

$$2 \times 2 = 3 + 1.$$

This simple check on total dimensionality sets the pattern for more complicated product spaces examined below.

The 4×4 representation of the rotation operator is said to be a *reducible* representation: it can be reduced to a *sum* of smaller dimensional representations. An *irreducible* representation is one in which there are no subspaces invariant under all rotations.

Recall that we constructed the reducible 4×4 representation by taking a direct product of the 2×2 spin one-half representations of the rotation group. The equation $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$ we used above to describe the ket spaces equivalently describes the rotation group representations within those subspaces.

One might wonder why we would bother to build two different bases for the same vector space. The reason is that different problems need different bases. For a system of two spins in an external magnetic field, not interacting with each other, the independent spins basis $|\uparrow\uparrow\rangle$, etc., is natural. On the other hand, for a hydrogen atom in *no* external field, but including an interaction between the spins (which are aligned with the magnetic dipole moments of the particles) the $|j, m\rangle$ basis is the right one: the interaction Hamiltonian is proportional to $\vec{S}^e \cdot \vec{S}^p$, which can be written $\frac{1}{2}(S_x^e + iS_y^e)(S_x^p - iS_y^p) + \frac{1}{2}(S_x^e - iS_y^e)(S_x^p + iS_y^p) + S_z^e S_z^p$, where we recognize the raising and lowering operators for the individual spins. This means that the state $|\uparrow\downarrow\rangle$, for example, cannot be an eigenstate if the spin term in the Hamiltonian is $\vec{S}^e \cdot \vec{S}^p$, but the states $|j, m\rangle$ are eigenstates because $\vec{S}^e \cdot \vec{S}^p$ commutes with the *total* angular momentum and its components.

But what would be a good basis for a hydrogen atom, including the $\vec{S}^e \cdot \vec{S}^p$ term, *and* in an external magnetic field? That is a nice exercise for the reader.

Adding a Spin to an Orbital Angular Momentum

In this section, we consider a hydrogen atom in a state with nonzero *orbital* angular momentum, $\vec{L} \neq 0$. Such orbital motion is equivalent to an electric current loop and generates a magnetic field. The magnetic dipole moment associated with the electron spin interacts with this field, the appropriate Hamiltonian having a term proportional to $\vec{L} \cdot \vec{S}$, and is termed the spin-orbit interaction. The proton also has a magnetic moment, but that is three orders of magnitude smaller than the electron's, so we'll neglect it for now.

The spin-orbit interaction $\vec{L} \cdot \vec{S}$ is most naturally analyzed in the basis states of *total* angular momentum, $|j, m\rangle$, where $\vec{J} = \vec{L} + \vec{S}$ (see the analogous discussion of the spin-spin interaction above). Write the orbital angular momentum eigenstates $|l, m_l\rangle$ and the spin states $|s, m_s\rangle$ where $|\frac{1}{2}, \frac{1}{2}\rangle = |\uparrow\rangle$ and $|\frac{1}{2}, -\frac{1}{2}\rangle = |\downarrow\rangle$. The product space $|l, m_l\rangle \otimes |s, m_s\rangle$ is $2(2l+1)$ dimensional: a single ket in this product space would be fully described by $|l, m_l; s, m_s\rangle$, but since both l, s are

constant throughout the problem, the only actual *variables* are m_l, m_s so we'll write the ket in the more compact form $|m_l, m_s\rangle_{m_l m_s}$, for example $|2, \frac{1}{2}\rangle_{m_l m_s}$.

The maximum possible angular momentum component in the z -direction is clearly $(l + \frac{1}{2})\hbar$, for the state $|l, \frac{1}{2}\rangle_{m_l m_s}$. In the *total* angular momentum representation, this must be the state $|j, m\rangle = |l + \frac{1}{2}, l + \frac{1}{2}\rangle_{jm}$. So the two different bases have a common member:

$$|l + \frac{1}{2}, l + \frac{1}{2}\rangle_{jm} = |l, \frac{1}{2}\rangle_{m_l m_s}.$$

In the total angular momentum $|j, m\rangle$ representation, $|l + \frac{1}{2}, l + \frac{1}{2}\rangle_{jm}$ is the top m state of a multiplet having $2(l + \frac{1}{2}) + 1 = 2l + 2$ members. The proof is the same as that for adding two spins one-half: the total spin raising operator gives zero acting on this state. And, just as for the spin-spin case, the next member down of the multiplet is generated by applying the lowering operator:

$$\begin{aligned} J_- |l + \frac{1}{2}, l + \frac{1}{2}\rangle_{jm} &= \sqrt{2l + 1} \hbar |l + \frac{1}{2}, l - \frac{1}{2}\rangle_{jm} \\ &= (L_- + S_-) |l, \frac{1}{2}\rangle_{m_l m_s} \\ &= \sqrt{2l} \hbar |l - 1, \frac{1}{2}\rangle_{m_l m_s} + \hbar |l, -\frac{1}{2}\rangle_{m_l m_s}. \end{aligned}$$

Therefore

$$|l + \frac{1}{2}, l - \frac{1}{2}\rangle_{jm} = \sqrt{\frac{2l}{2l + 1}} |l - 1, \frac{1}{2}\rangle_{m_l m_s} + \sqrt{\frac{1}{2l + 1}} |l, -\frac{1}{2}\rangle_{m_l m_s}.$$

This state $|l + \frac{1}{2}, l - \frac{1}{2}\rangle_{jm}$ lies in the $m = l - \frac{1}{2}$ subspace, which is two-dimensional, having basis vectors $|l - 1, \frac{1}{2}\rangle_{m_l m_s}$ and $|l, -\frac{1}{2}\rangle_{m_l m_s}$ in the $| \rangle_{m_l m_s}$ representation. So it must have two basis vectors in the $| \rangle_{jm}$ representation as well. The other $| \rangle_{jm}$ ket must be orthogonal to $|l + \frac{1}{2}, l - \frac{1}{2}\rangle_{jm}$ and normalized: it can only be

$$|l - \frac{1}{2}, l - \frac{1}{2}\rangle_{jm} = \sqrt{\frac{2l}{2l + 1}} |l, -\frac{1}{2}\rangle_{m_l m_s} - \sqrt{\frac{1}{2l + 1}} |l - 1, \frac{1}{2}\rangle_{m_l m_s}.$$

We've represented this new ket in $| \rangle_{jm}$ as the top state of a $j = l - \frac{1}{2}$ multiplet. It's easy to check that this is indeed the case: it has $m = l - \frac{1}{2}$, and J_+ acting on it gives zero, so it has to be the top member of its multiplet.

The only ambiguity is an overall phase: the **Condon-Shortley convention** is that *the highest m -state of the larger component angular momentum is assigned a positive coefficient*.

So $|l - \frac{1}{2}, l - \frac{1}{2}\rangle_{jm}$ is the top state of a new multiplet having $2(l - \frac{1}{2}) + 1 = 2l$ members. The two multiplets $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$ taken together have $2(2l + 1)$ members, and therefore span the whole $2(2l + 1)$ dimensional space. The rest of the $| \rangle_{jm}$ basis vectors are generated by repeated application of the lowering operator in the two multiplets.

The reason there are only two multiplets in this problem is that there are only two ways the spin one-half can point relative to the orbital angular momentum. Recalling that for the two spins we expressed the product space a sum of a spin 1 space and a spin 0 space, $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$, the analogous equation here is

$$\frac{1}{2} \otimes l = (l + \frac{1}{2}) \oplus (l - \frac{1}{2}).$$

For the general case of adding angular momenta j_1, j_2 with $j_1 \geq j_2$, $2j_2 + 1$ multiplets are generated, corresponding to the number of possible relative orientations of the two angular momenta.

Adding Two Angular Momenta: the General Case

The space of kets describing two angular momenta j_1, j_2 is the direct product of two spaces each for a single angular momentum, but the direct product nature of the kets is usually not made explicit, for example $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$ is usually written as a single ket $|j_1, m_1; j_2, m_2\rangle$. Just as in the examples above, since j_1, j_2 are fixed throughout, they don't need to be written into every ket, we'll just denote the ket by $|m_1, m_2\rangle$, or, when dealing with numerical values, append $m_1 m_2$ as a suffix, for example $|2, 3\rangle_{m_1 m_2}$.

The kets $|m_1, m_2\rangle$ form a *complete orthonormal basis* of the $(2j_1 + 1)(2j_2 + 1)$ dimensional product space of the two angular momenta: they are the eigenstates of the complete set of commuting variables $\vec{J}_1^2, J_{1z}, \vec{J}_2^2, J_{2z}$.

Total Angular Momentum Basis States

There is of course an alternative complete orthogonal basis of the space of the two angular momenta: for total angular momentum $\vec{J} = \vec{J}_1 + \vec{J}_2$, a different set of complete commuting variables is: $\vec{J}_1^2, \vec{J}_2^2, \vec{J}^2, J_z$. (This is *not* the same set of states as in the previous paragraph: for example, \vec{J}^2 does *not* commute with J_{1z} . Check it out!) We shall establish later in the lecture that the allowed values of total angular momentum range from $j = j_1 + j_2$ to $j = |j_1 - j_2|$, just as one would naively expect.

This alternative set is a better basis set for two angular momenta interacting with each other—an interaction term like $\vec{J}_1 \cdot \vec{J}_2$ can change m_1, m_2 but not $m = m_1 + m_2$, or \vec{J}^2 .

As always, we're taking \vec{J}_1^2, \vec{J}_2^2 to be constants throughout, so the significant variables here are \vec{J}^2 and J_z , and we write the states simply as $|j, m\rangle$ or when we have numerical values, $|3, 1\rangle_{jm}$, following the notation introduced above. Of course, $\vec{J}^2 |j, m\rangle = j(j+1)\hbar^2 |j, m\rangle$, and $J_z |j, m\rangle = m\hbar |j, m\rangle$.

Going from One Basis to the Other: the Clebsch-Gordan Coefficients

How do we write a state $|j, m\rangle$ in terms of the states $|m_1, m_2\rangle$? Furthermore, how do we prove the new set of states $|j, m\rangle$ is a complete basis for the space?

We know that the set of states $|m_1, m_2\rangle \equiv |j_1, m_1\rangle \otimes |j_2, m_2\rangle$ is a complete basis, since the whole space is a product space of the j_1 and j_2 spaces, which are spanned by the sets $|m_1\rangle, |m_2\rangle$ respectively. Therefore, the identity operator can be written

$$I = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |m_1, m_2\rangle \langle m_1, m_2|.$$

It follows that any total angular momentum eigenket $|j, m\rangle$ can be expressed as a sum over the basis vectors $|m_1, m_2\rangle$:

$$|j, m\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |m_1, m_2\rangle \langle m_1, m_2 | j, m\rangle$$

The coefficients $\langle m_1, m_2 | j, m\rangle$ are called the *Clebsch Gordan coefficients*, often written CG coefficients.

One immediate property of the CG coefficients is that $\langle m_1, m_2 | j, m\rangle = 0$ unless $m = m_1 + m_2$. This follows from the operator identity $J_z = J_{1z} + J_{2z}$ taken between a bra and a ket from different bases,

$$\langle m_1, m_2 | J_z | j, m\rangle = \langle m_1, m_2 | J_{1z} + J_{2z} | j, m\rangle$$

and

$$J_z | j, m\rangle = m\hbar | j, m\rangle, \quad \langle m_1, m_2 | (J_{1z} + J_{2z}) = \langle m_1, m_2 | (m_1 + m_2)\hbar,$$

so

$$(m - m_1 - m_2) \langle m_1, m_2 | j, m \rangle = 0.$$

We already know that the maximum value of m_1 is j_1 , and of m_2 is j_2 , so the maximum value of m is $j_1 + j_2$. Therefore, the maximum value of $j = j_1 + j_2$, because if it could go any higher, there would be a higher m somewhere in the space, contradicting $m = m_1 + m_2$.

For the set $|m_1, m_2\rangle$, there is one ket having this maximal value of m : $|j_1, j_2\rangle_{m_1 m_2}$.

Equally, in the set of states $|j, m\rangle$ there is only one with the maximal m : $|j_1 + j_2, j_1 + j_2\rangle_{jm}$.

Therefore, these two kets must be identical (setting the arbitrary phase factor equal to one):

$$|j_1, j_2\rangle_{m_1 m_2} = |j_1 + j_2, j_1 + j_2\rangle_{jm}.$$

Now $|j_1 + j_2, j_1 + j_2\rangle_{jm}$ is the top ket in a multiplet having $2j + 1 = 2(j_1 + j_2) + 1$ members.

The next-to-top member of the multiplet is generated as before by applying the lowering operator to both representations:

$$J_- |j_1 + j_2, j_1 + j_2\rangle_{jm} = (J_{1-} + J_{2-}) |j_1, j_2\rangle_{m_1 m_2}$$

giving

$$\sqrt{2(j_1 + j_2)\hbar} |j_1 + j_2, j_1 + j_2 - 1\rangle_{jm} = \sqrt{2j_1\hbar} |j_1 - 1, j_2\rangle_{m_1 m_2} + \sqrt{2j_2\hbar} |j_1, j_2 - 1\rangle_{m_1 m_2}$$

so

$$|j_1 + j_2, j_1 + j_2 - 1\rangle_{jm} = \sqrt{\frac{j_1}{j_1 + j_2}} |j_1 - 1, j_2\rangle_{m_1 m_2} + \sqrt{\frac{j_2}{j_1 + j_2}} |j_1, j_2 - 1\rangle_{m_1 m_2}$$

and by exact analogy with the spin orbit case, the other $| \rangle_{jm}$ basis state in the $m = j_1 + j_2 - 1$ subspace is

$$|j_1 + j_2 - 1, j_1 + j_2 - 1\rangle_{jm} = -\sqrt{\frac{j_2}{j_1 + j_2}} |j_1 - 1, j_2\rangle_{m_1 m_2} + \sqrt{\frac{j_1}{j_1 + j_2}} |j_1, j_2 - 1\rangle_{m_1 m_2}$$

with the appropriate (Condon Shortley) sign convention for $j_1 > j_2$. This is the top member of a multiplet having $j = j_1 + j_2 - 1$, and so $2(j_1 + j_2 - 1) + 1 = 2(j_1 + j_2) - 1$ members (checked as usual by applying J_+ and getting zero).

To proceed further, the lowering operator is applied once more, to enter the $m = j_1 + j_2 - 2$ subspace. In the $|\rangle_{m_1 m_2}$ representation, this has *three* independent basis vectors (provided $j_2 > \frac{1}{2}$): $|j_1 - 2, j_2\rangle_{m_1 m_2}$, $|j_1 - 1, j_2 - 1\rangle_{m_1 m_2}$, $|j_1, j_2 - 2\rangle_{m_1 m_2}$. But only two kets have been lowered in the $|\rangle_{jm}$ representation—the missing third $|\rangle_{jm}$ ket in the $m = j_1 + j_2 - 2$ subspace must be the top member of *another* new multiplet having $j = j_1 + j_2 - 2$, and so $2(j_1 + j_2) - 3$ members.

Note that the coefficients generated by the lowering operators are all real, so all three $|\rangle_{jm}$ kets in the $m = j_1 + j_2 - 2$ subspace can be written in terms of the $|\rangle_{m_1 m_2}$ kets with real coefficients.

This process can be repeated until the $|\rangle_{jm}$ multiplets generated span the space. Recall that the dimensionality of the space, from the $|\rangle_{m_1 m_2}$ representation, is $(2j_1 + 1)(2j_2 + 1)$. The multiplets in $|\rangle_{jm}$ add to a total dimensionality

$$2(j_1 + j_2) + 1 + 2(j_1 + j_2) - 1 + 2(j_1 + j_2) - 3 + \dots$$

but where do we stop? Common sense suggests that for $j_1 > j_2$, the minimum total angular momentum must be $j = j_1 - j_2$. Common sense is not necessarily to be trusted, but it is clear that all the members of the multiplets in $|\rangle_{jm}$ generated by using the lowering operator, followed by introducing a new orthogonal multiplet top member each time, as described above, are independent orthonormal kets, and if we stop at $j = j_1 - j_2$, the total number generated is

$$\sum_{n=|j_1-j_2|}^{j_1+j_2} (2n+1) = (2j_1+1)(2j_2+1).$$

(Use $\sum_{n=0}^m (2n+1) = (m+1)^2$.) This establishes that including all total angular momenta between $|j_1 - j_2|$ and $j_1 + j_2$ does in fact give a complete basis spanning the space, so

$$j_1 \otimes j_2 = (j_1 + j_2) \oplus (j_1 + j_2 - 1) \oplus \dots \oplus (|j_1 - j_2|).$$

Calculating Clebsch-Gordan Coefficients Using Recursion Relations

The scheme presented above, constructing a succession of multiplets beginning from the highest m state and using the Condon-Shortley convention to settle signs, will generate all the CG coefficients. However, another approach proves useful in later work. Recall that by finding

matrix elements of $J_z = J_{1z} + J_{2z}$ between a $\langle m_1, m_2 |$ bra and a $|j, m\rangle$ ket, we established that the Clebsch-Gordan coefficients are zero unless $m = m_1 + m_2$. A parallel evaluation of matrix elements of $J_{\pm} = J_{1\pm} + J_{2\pm}$ yields a relationship between three CG coefficients:

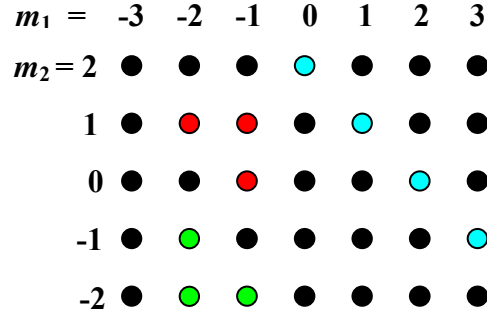
$$\langle m_1, m_2 | J_{+} | j, m \rangle = \langle m_1, m_2 | J_{1+} | j, m \rangle + \langle m_1, m_2 | J_{2+} | j, m \rangle$$

yields

$$\begin{aligned} \sqrt{j(j+1) - m(m+1)} \langle m_1, m_2 | j, m+1 \rangle = \\ \sqrt{j_1(j_1+1) - m_1(m_1-1)} \langle m_1-1, m_2 | j, m \rangle + \sqrt{j_2(j_2+1) - m_2(m_2-1)} \langle m_1, m_2-1 | j, m \rangle \end{aligned}$$

where J_{1+} acting to the left *reduces* m_1 by one. (Here, obviously, we must choose $m = m_1 + m_2 - 1$ to have nonzero coefficients.)

To visualize what's going on with all these coefficients, remember m_1 can take $2j_1 + 1$ values and m_2 can take $2j_2 + 1$ values, so for given j_1, j_2 every possible state of the two spins can be represented by a dot on a $(2j_1 + 1) \times (2j_2 + 1)$ grid: here's $j_1 = 3, j_2 = 2$:



Each dot represents an $|m_1, m_2\rangle$ state for $j_1 = 3, j_2 = 2$.

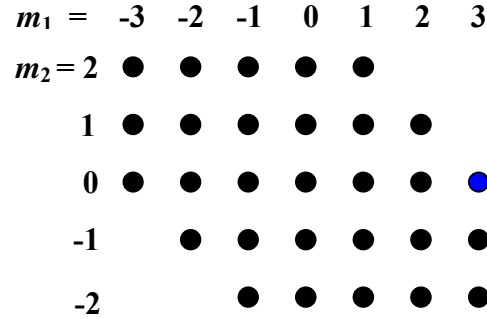
The four blue dots have $m = m_1 + m_2 = 2$.

For given total j , the coefficients for the three red dots are connected by a J_{+} recursion relation.

How do these dots relate to the CG coefficients? the top right-hand dot (3, 2) uniquely represents the $j = 5, m = 5$ state of total angular momentum. The next dots down, (2, 2) and (3, 1), correspond to two CG coefficients for $j = 5$ and two *different* CG coefficients for $j = 4$.

If we now pick one value of j less than $j_1 + j_2$, each dot in the grid will correspond to one coefficient.

Note that having fixed j , the grid will be curtailed: let's take $j = 3$, so $m = m_1 + m_2 = 3$ at most. Then the grid loses its far corners:



Nonzero grid members for $j = 3$.

Let us examine for this fixed j which CG coefficients are where in this curtailed grid.

There are a total of $2j + 1 = 7$ states for $m = 3, 2, \dots, -3$.

The top state, $j = 3, m = 3$, or $|3, 3\rangle_{jm}$, is given by three coefficients on the top diagonal line (it's in a three-dimensional subspace, and orthogonal to $j = 5$ and $j = 4$ multiplet members $|5, 3\rangle_{jm}, |4, 3\rangle_{jm}$ which are also in the $m = 3$ subspace). We're not at this point *calculating* these coefficients, we're just trying to find them a home.

Applying the lowering operator to $|3, 3\rangle_{jm}$ gives a vector in the four-dimensional $m = 2$ subspace, the coefficients would belong to the next diagonal down, which has four elements. (This subspace also includes the top member of the $j = 2$ multiplet.) Using the lowering operator one more time we enter the five-dimensional $m = 1$ subspace—but that is the maximum number of dimensions in this problem, since angular momenta 3 and 2 cannot be added to give a $j = 0$ scalar.

Having now, for this particular j made from $j_1 + j_2$, found where all the CG coefficients for all the $2j + 1$ multiplet members are located, we shall see how they can all be systematically calculated using the recursion relations generated by $J_{\pm} = J_{1\pm} + J_{2\pm}$.

We've mapped the recursion relations on the diagram: given j, j_1, j_2 the three red dots at $(m_1, m_2), (m_1 - 1, m_2), (m_1, m_2 - 1)$ (with $m_1 = -1, m_2 = 1$ in this example) locate the three CG coefficients satisfying the linear equation above from

$$\langle m_1, m_2 | J_+ | j, m \rangle = \langle m_1, m_2 | J_{1+} | j, m \rangle + \langle m_1, m_2 | J_{2+} | j, m \rangle$$

so if two of them are known the third is given. Similarly, the parallel equation generated by $J_- = J_{1-} + J_{2-}$ links the three green dots, at (m_1, m_2) , $(m_1 + 1, m_2)$, $(m_1, m_2 + 1)$.

We begin the computation of the CG coefficients with the blue dot $(m_1 = 3, m_2 = 0)$, the point on the leading “arrow” edges. Let us arbitrarily assign a value 1 to this point. If we make it the top member of a “green” triangle, that will link it to the dot below and to a dot to the right which is *off the array*. The dot off the array makes zero contribution, so we have an equation giving the value of the coefficient at the dot below the blue dot as a multiple of the value on the blue dot. We can then continue down to the next dot. We could instead have gone *up* from the blue dot using incomplete red triangles—in fact we can continue around the edge of the whole array. Then, once the values along the edges are fixed, the recursion triangles can be used to move inward and find the rest.

The point of this section is to establish that, apart from an overall multiplicative constant that must be fixed by normalization, *all* the CG coefficients for this value of j can be found *from the recursion relations alone*. The reason this is important is because the *same* algebraic structure, and therefore the same recursion relations, are used to define spherical tensors, so they can *also* be combined using the same CG coefficients. (We still need a sign convention here to present a complete table: so far, the different values of total j have arbitrary relative phases.)