Quasi-Perturbative RG-motivated Non-power expansions in QCD

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Subject and Motivation

In QFT we use Feynman PT expansion in $(\alpha)^n$ as a launch pad for RenormGroup powers of $\bar{\alpha}_s$ series singular at $\sim \mathbf{1}\mathrm{GeV}$. Meanwhile, it is known that:

– In the complex α plane, the origin $\alpha = 0$ is a singular point

– $\sum_n^\infty c_n \alpha^n$ is divergent as $c_n \sim n!$ at $n \gg 1$.

– For finite sum $\sum_n^N c_n \alpha^n$, value $n^* \sim 1/\alpha$ is a critical one. Here, power PT expansion can start to explode.

Subject and Motivation, II

– In the complex α plane the origin $\alpha = 0$ is a singular point,

– Expansion $\sum_n c_n \alpha^n$ is divergent as $c_n \sim n!$,

– Value $n^* \sim 1/\alpha$ is a critical one for $\sum_n^N c_n \alpha^n$.

Here, pQCD expansion does start to explode.

– In low-energy QCD, $n^* \sim 3 - 5$ and value of 3-, 4-loop calculation is under question. Examples are known.

How serious is this menace for practical use in low-energy pQCD calculation ?

(Sorry for **Terazawa et others 1965-??**)

Topics

1. – General comment on the α analyticity in QFT,
– RG+Källen-Lehmann $\to exp(-\frac{1}{\beta_0\,\alpha})$

2. Outline of ghost-free Analytic Perturbat Theory (APT)

- $\int$ transformations Euclid $\Leftrightarrow$ Minkow $\Leftrightarrow$ Distance;
- Ghost–, parameter– free Minkowskian $\alpha_M(s)$, Euclidean $\alpha_E(Q^2)$, and Distance $\alpha_D(r^2)$ QCD couplings;
- Non-power sets $\{\mathcal{A}_k\}$, $\{\mathfrak{A}_k\}$, $\{\aleph_k\}$ in QCD

3. – Mapping $\mathcal{A}_k, \mathfrak{A}_k, \aleph_k \longrightarrow w_k^R(\alpha_s)$, $[R = E, M, D]$,
– RG-inspired expansion functions $(\alpha_s)^k \to w_k^R(\alpha_s)$,
– Quasi-perturbative expansions over sets $\{w_k^R(\alpha_s)\}$.

Preamble, I

History : Essential singularity at the origin $\alpha = 0$ of expansion parameter $z = \alpha$ complex plane is an indispensable feature of systems with large number degrees of freedom. The type of singularity can be found by saddle point method for functional integral or (e.g., in QFT) by combination of causality and renormalization invariance.

Preamble, II

A P T : In pQCD, one uses expansions in powers **of the same** $\bar{\alpha}_s$, i.e., in $[\bar{\alpha}_{\mathbf{s}}(\mathbf{x})]^{\mathbf{k}}$; $(\mathbf{x}=\mathbf{Q^2}, \mathbf{s}, \mathbf{r^{-2}})$ at Euclidean (Q^2), Minkowskian (s) or Distance (r^{-2}) pictures. Instead,

in Analytic Perturbation Theory, one has:

– Different ghost-free couplings: $\alpha_{\mathbf{E}}(\mathbf{Q^2})$, $\alpha_{\mathbf{M}}(\mathbf{s})$, $\alpha_{\mathbf{D}}(\mathbf{r^{-2}})$

– Non-power functional expansions :

$$\mathbf{D}(\mathbf{Q^2}) = \mathbf{1} + \mathbf{d_1}\alpha_{\mathbf{E}}(\mathbf{Q^2}) + \mathbf{d_2}\,\mathcal{A}_{\mathbf{2}}(\mathbf{Q^2}) + \mathbf{d_3}\,\mathcal{A}_{\mathbf{3}}(\mathbf{Q^2}) + \ldots\,;$$

$$\mathbf{R}(\mathbf{s}) = \mathbf{1} + \mathbf{d_1}\,\alpha_{\mathbf{M}}(\mathbf{s}) + \sum_{\mathbf{k}\geq\mathbf{2}} \mathbf{d_2}\,\mathfrak{A}_{\mathbf{k}}(\mathbf{s})\,; \qquad \mathbf{V}(\mathbf{r}) \sim \sum_{\mathbf{k}} \mathbf{v_k}\,\aleph_{\mathbf{k}}(\mathbf{r})$$

over sets $\{\mathcal{A}_{\mathbf{k}}\}\,,\{\mathfrak{A}_{\mathbf{k}}\}\,,\{\aleph_{\mathbf{k}}\}$ related $\mathfrak{A}_{\mathbf{k}} \leftrightarrow \mathcal{A}_{\mathbf{k}} \leftrightarrow \aleph_{\mathbf{k}}$ by integral transformations. This is important below 10 GeV .

Preamble, III

NEW :

- The APT non-power sets $\{\mathcal{A}_k\}$, $\{\mathfrak{A}_k\}$, $\{\aleph_k\}$ – by transition $\Lambda \to (\mu,\, \alpha_\mu)$ parameterization induce expansion sets $\{w_k^E(\alpha)\}$, $\{w_k^M(\alpha)\}$, $\{w_k^D(\alpha)\}$ which replace common power set $\{\alpha^k\}$.
- These new expansions over non-power sets $\{w_k^R(\alpha)\}$ can be used more widely than usual RG-improved PT; e.g., for multi-variable objects.
- Analogous to APT series for $D(Q^2)$, $R(s)$, $V(r)$, new expansions over $\{w_k^R(\alpha)\}$ reveal much better convergence properties.

1. History - Non-analyticity in α

- Dyson 1949 argument on essential singularity at origin of α complex plane
- RG + causality (Källen-Lehmann Q^2-analyticity) yields $\sim \exp[-1/(\beta_0\,\alpha_s)]$ for $\bar{\alpha}$ **(Sh. '76)**
- Saddle point method for functional integral for class of QFT models. **(Lipatov '76, Sh. '79, Kazakov-Sh '80)**

2. Main Results of APT, 1-loop case

RG defined QCD coupling $\bar{\alpha}_s(Q^2)$ – real function of real Q^2. It is sum of UV logs with ghost. At the 1-loop case

$$\bar{\alpha}_{\mathbf{s}}^{(\mathbf{1})}(\mathbf{Q^2}) = \frac{\alpha_{\mathbf{s}}}{\mathbf{1} + \beta_{\mathbf{0}}\,\alpha_{\mathbf{s}}\,\ln(\mathbf{Q^2}/\mu^{\mathbf{2}})} = \frac{\mathbf{1}}{\beta_{\mathbf{0}}\,\ln(\mathbf{Q^2}/\mathbf{\Lambda^2})}\,, = \frac{\mathbf{1}}{\beta_{\mathbf{0}}\,\mathbf{l}}$$

In APT, we have diverse ghost-free couplings

Minkow : $\bar{\alpha}_{\mathbf{M}}^{(\mathbf{1})}(\mathbf{s}) = \frac{\mathbf{1}}{\pi\,\beta_{\mathbf{0}}} \arccos \frac{\mathbf{L}}{\sqrt{\mathbf{L^2} + \pi^{\mathbf{2}}}}\Big|_{\mathbf{L>0}} = \frac{\mathbf{1}}{\pi\beta_{\mathbf{0}}} \arctan \frac{\pi}{\mathbf{L}}\,;$

Euclid : $\alpha_{\mathbf{E}}^{(\mathbf{1})}(\mathbf{Q^2}) = \frac{\mathbf{1}}{\beta_{\mathbf{0}}}\left[\frac{\mathbf{1}}{\ell} - \frac{\mathbf{\Lambda^2}}{\mathbf{Q^2} - \mathbf{\Lambda^2}}\right]\,;$ $\qquad \mathbf{L} = \ln\frac{\mathbf{s}}{\mathbf{\Lambda^2}}\,.$

Distance: $\bar{\alpha}_{\mathbf{D}}(\mathbf{r}) = \mathbb{F}\,[\alpha_{\mathbf{E}}]\,(\mathbf{r})$ ($\mathbb{F}$ - Fourier tr.) – see Fig.1

Comparing $\bar{\alpha}_s$ with α_E and α_M

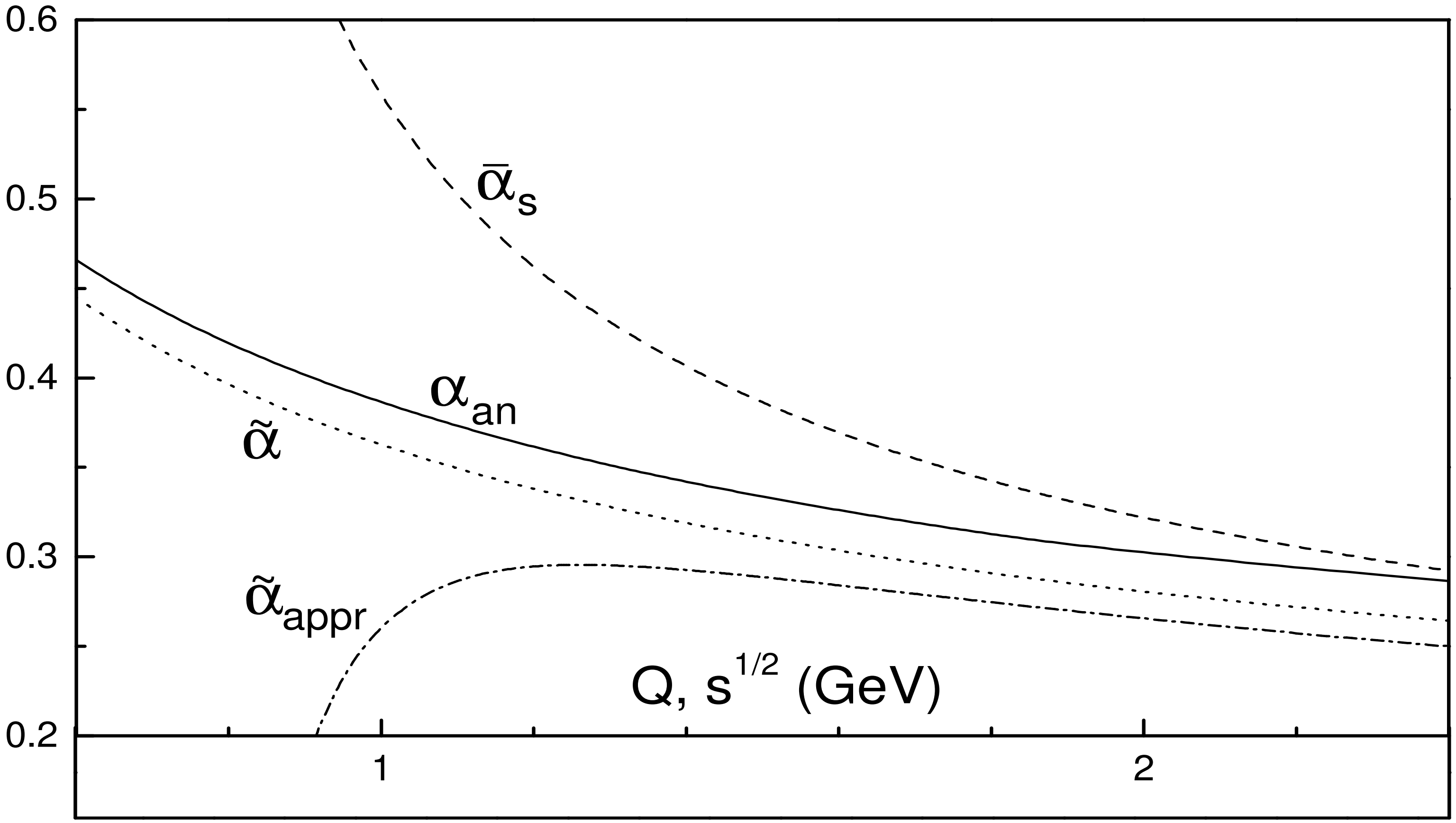

Figure 1: *Comparison of usual QCD coupling $\bar{\alpha}_s$ with Euclidean $\alpha_E = \alpha_{an}$ and Minkowskian $\alpha_M = \tilde{\alpha}$ ones in a few* GeV *region.*

Main Results of APT. 1-loop case. Cont'd

Higher functions $\mathcal{A}_k$, $\mathfrak{A}_k$, $\aleph_k$ are defined via $\mathcal{A}_1 = \alpha_E$, $\mathfrak{A}_1 = \alpha_M$, $\aleph_1 = \alpha_D$ by differ. recursive relations

$$\frac{1}{\mathbf{k}}\frac{\mathbf{d}\mathcal{A}^{(1)}_{\mathbf{k}}}{\mathbf{dl}} = -\,\beta_0\,\mathcal{A}^{(1)}_{\mathbf{k+1}}(\mathbf{l}), \quad \frac{1}{\mathbf{k}}\frac{\mathbf{d}\mathfrak{A}^{(1)}_{\mathbf{k}}(\mathbf{L})}{\mathbf{dL}} = -\,\beta_0\,\mathfrak{A}^{(1)}_{\mathbf{k+1}}\,, \ldots .$$

The 2nd and 3rd one-loop functions are

$$\mathcal{A}^{(1)}_{\mathbf{2}}(\mathbf{l}) = \frac{1}{\beta_0^2}\left(\frac{1}{\mathbf{l}^2} - \frac{\mathbf{e}^{\mathbf{l}}}{(\mathbf{e}^{\mathbf{l}}-1)^2}\right), \quad \mathfrak{A}^{(1)}_{\mathbf{2}}(\mathbf{L}) = \frac{1}{\beta_0^2}\,\frac{1}{\mathbf{L}^2+\pi^2}\,;$$

$$\mathcal{A}_{\mathbf{3}}(\mathbf{l}) = \frac{1}{\beta_0^3}\left(\frac{1}{\mathbf{l}^3} - \frac{1}{2}\,\frac{\mathbf{e}^{\mathbf{l}}+\mathbf{e}^{2\mathbf{l}}}{(\mathbf{e}^{\mathbf{l}}-1)^3}\right), \quad \mathfrak{A}_{\mathbf{3}}(\mathbf{L}) = \frac{1}{\beta_0^3}\,\frac{\mathbf{L}}{(\mathbf{L}^2+\pi^2)^2}\,,$$

which are not powers of $\alpha^{(1)}_E(s)$ and $\alpha^{(1)}_M(s)$.

Non-power expansions for QCD observables

APT couplings are related by integral operation

$$\alpha_E(Q^2) = Q^2 \int_0^\infty \frac{\bar{\alpha}_M(s)\, ds}{(s+Q^2)^2} \equiv \mathbb{D}\left[\bar{\alpha}_M\right](Q^2) \qquad (\mathbb{D})$$

= **integral transformation** $\mathbb{D}$. Analogous transform-s (Fourier, $\mathbb{F}$) relates functions in various pictures

$$\mathfrak{A}_k(s) \to \mathcal{A}_k(Q^2) \equiv \mathbb{D}\left[\mathfrak{A}_k\right](Q^2), \quad \aleph_k(r) = \mathbb{F}\left[\mathcal{A}_k\right](r)$$

$$d(Q^2) = d_1 \alpha_E(Q^2) + d_2\, \mathcal{A}_2(Q^2) + d_3\, \mathcal{A}_3(Q^2) + \ldots\,,$$

$$r(s) = \sum_k d_k\, \mathfrak{A}_k(s)\,, \qquad V(r) = \sum_k v_k\, \aleph_k(r)\,.$$

The whole APT construction is “picture"-invariant ,

while common PT+RG $\sim$ pQCD **is not.**

$\int$ Transformations : $\bar{\alpha}_s(Q^2) \to \bar{\alpha}_E(Q^2) \Leftrightarrow \bar{\alpha}_M(s)$

Contour integration [Radiyshkin-Krasnikov-Pivovarov '82]

$$\bar{\alpha}_s(Q^2) \to \frac{i}{2\,\pi}\int_{s-i\varepsilon}^{s+i\varepsilon}\frac{dz}{z}\,\bar{\alpha}_s(-z) = \bar{\alpha}_M(s) \equiv \mathbb{R}\left[\bar{\alpha}_s\right](s)$$

yields ghost-free expression with π^2 terms summed.
Reverse transformation [Milton,Solovtsov + D.Sh. '97-'98]

$\mathbb{D} = [\mathbb{R}]^{-1}$ yields non-singular expression,

$$\bar{\alpha}_s^{(1)}(Q^2) = \frac{1}{\beta_0\,\ell} \to \mathbb{R}\left[\bar{\alpha}_s\right] = \bar{\alpha}_M^{(1)}(s) = \frac{1}{\beta_0\,\pi}\arctan\left(\frac{\pi}{L}\right) \to$$

$$\to \mathbb{D}[\bar{\alpha}_M^{(1)}] = \frac{1}{\beta_0}\left[\frac{1}{\ell} + \frac{\Lambda^2}{\Lambda^2 - Q^2}\right] \equiv \bar{\alpha}_E^{(1)}(Q^2)\,.$$

Logic of APT scheme is displayed at Fig.2

Minkowskian vs. Euclidean

Minkowskian and Euclidean ghost-free functions are related by $\mathbb{D}$, $\mathbb{R}$ transformations: 1-loop case

$$\mathbb{D}\left[\frac{1}{\pi}\arccos\frac{L}{\sqrt{L^2+\pi^2}}\right] = \frac{1}{\ln(Q^2/\Lambda^2)} - \frac{\Lambda^2}{Q^2-\Lambda^2}\,,$$

$$\mathbb{R}\left[\frac{1}{\ln^2(Q^2/\Lambda^2)} - \frac{Q^2\Lambda^2}{(Q^2-\Lambda^2)^2}\right] = \frac{1}{[L^2+\pi^2]}\,,$$

Generally, $\mathcal{A}_k(Q^2) = \mathbb{D}\left[\mathfrak{A}_k\right]$, $\mathfrak{A}_k(s) = \mathbb{R}\left[\mathcal{A}_k\right]$, $\aleph_k(r^{-2}) = \mathbb{F}\left[\mathcal{A}_k\right]$.

All the construction is "transformation-invariant" :

$$\mathbb{D}\left[R(s) = \sum_k d_k\,\mathfrak{A}_k(s)\right] \Rightarrow D(Q^2) = \sum_k d_k\,\mathcal{A}_k(Q^2)\,.$$

gAPT – closed scheme

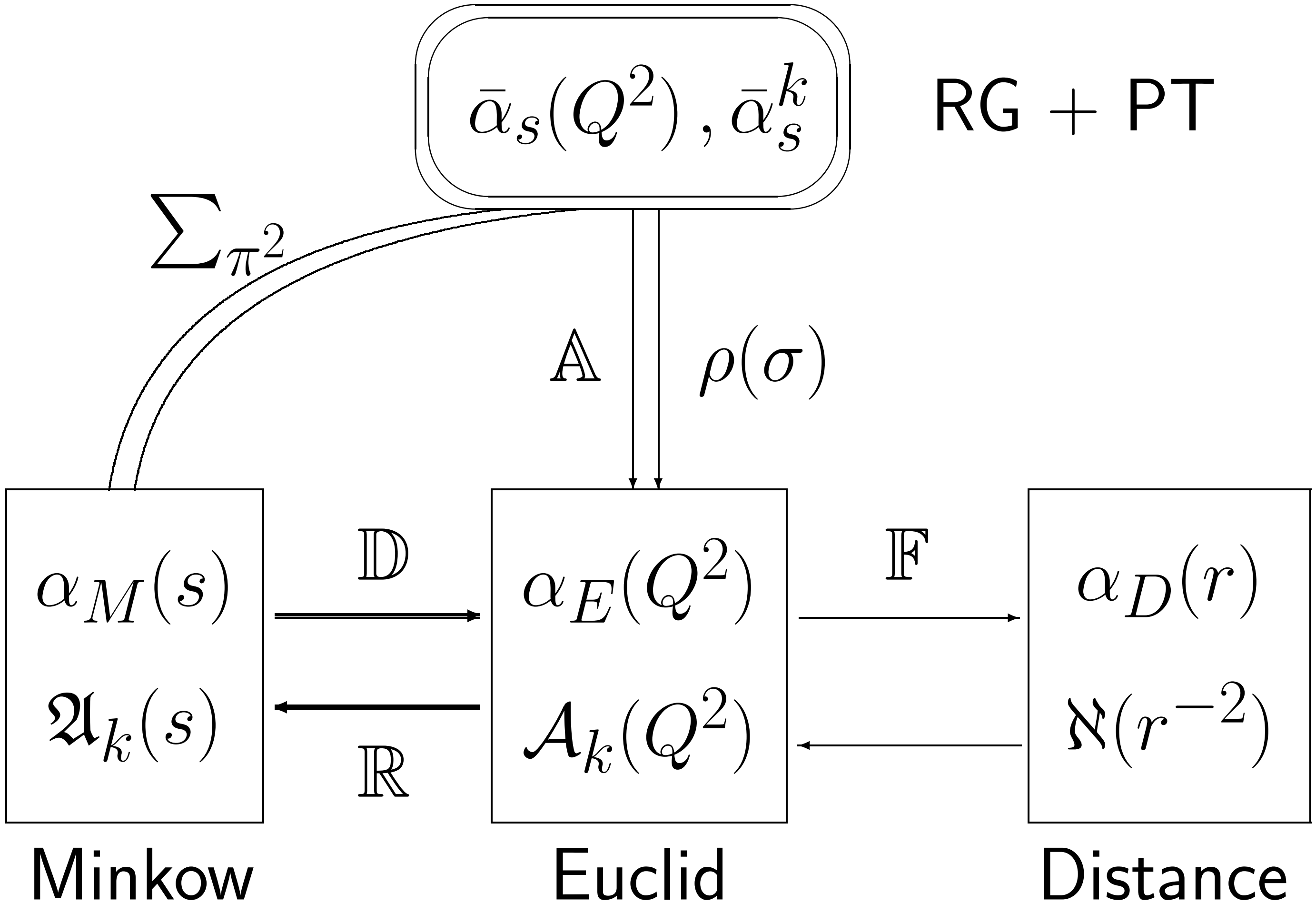

Fig.2 *Logic of APT scheme.*

Sketch of global APT algorithm

Elegant form of APT uses spectral density $\rho(\sigma)$ from PT input. Then all functions in 3 pictures are

$$\mathcal{A}_k = \frac{1}{\pi}\int\limits_0^\infty \frac{\rho_k(\sigma)\, d\sigma}{\sigma + Q^2}\,, \quad \mathfrak{A}_k = \frac{1}{\pi}\int\limits_s^\infty \frac{d\sigma}{\sigma}\,\rho_k\,, \; \aleph_k(r) = \frac{1}{\pi}\int\limits_0^\infty \frac{\rho_k\, d\sigma}{\sigma}\left(1 - e^{-r\sqrt{\sigma}}\right)\,.$$

1-loop case

$$\rho_1^{(1)} = \frac{1}{\beta_0\,[L_\sigma^2 + \pi^2]}\,; \qquad L_\sigma = \ln\frac{\sigma}{\Lambda^2}\,; \qquad k\,\beta_0\,\rho_{k+1}^{(1)} = -\frac{d\,\rho_k^{(1)}(\sigma)}{d\,L_\sigma}\,.$$

This was generalized for higher-loops case and real QCD with quark thresholds. This global APT [Sh.- 00/01] was successively used for data fitting.

Effect of Transition $\bar{\alpha}_s^k \to \{\mathcal{A}_k\}\ ,\ \{\mathfrak{A}_k\}\ ,\ \{\aleph_k\}$

To see effect of transition from $\left\{\bar{\alpha}_s^k\right\}$ to non-power $\{\mathcal{A}_k\}, \{\mathfrak{A}_k\}\ , \{\aleph_k\}\ ,$ look at their properties. Qualitatively, all 3 are similar:

I. They consist of ghost-free functions.

II. First ones, new couplings, $\alpha_E,\ \alpha_M,\ \alpha_D$

- ◇ are monotonic and IR finite $= 1/\beta_0 \simeq 1.4\,.$
- ◇ At the UV limit $\sim 1/\ln x \sim \bar{\alpha}_s(x)$, $(x = Q^2, s, r^{-2})\,.$
- ◇ 2nd and 3rd loop functions are close to each other = loop-stability.

$E \to M$ “distorting mirror” [Milton-Solovtsov '98]

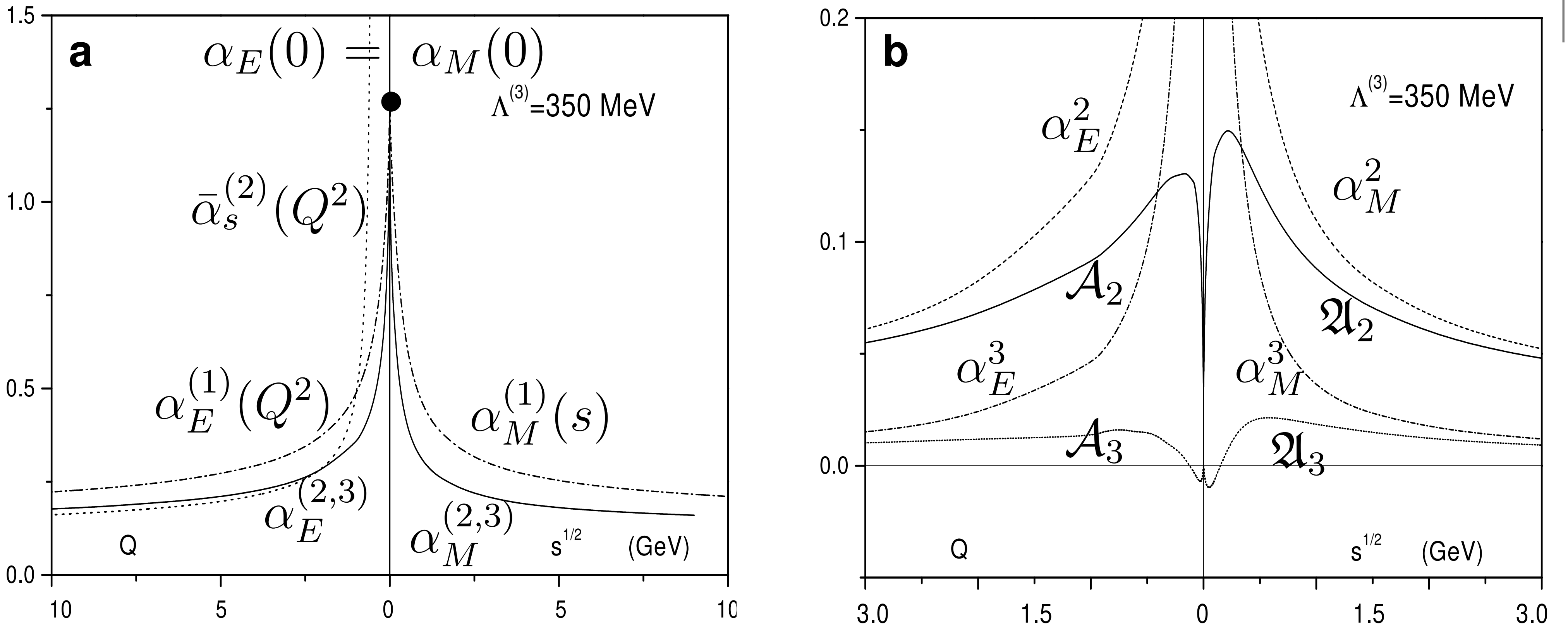

Figure 3: **a** – *Space-like and time-like APT couplings in* GeV *domain.* **b** – *“Distorted mirror symmetry” for global functions. All* **b** *curves – exact 2–loop solutions calculated via Lambert function.*

Effect of Transition - 2

III. All the other functions $(k \geq 2)$:

♡ start from zero $\mathcal{A}_k(0), \mathfrak{A}_k(0), \aleph_k\,(0) = 0;$

♡ in the UV limit $\sim 1/(\ln x)^k \sim \bar{\alpha}_s^k(x)$.

♡ 2nd, $\mathcal{A}_2$, $\mathfrak{A}_2$, $\aleph_2$ obey max at $x \sim \Lambda^2$.

♡ Higher ones $\mathcal{A}_{k\geq 3}$; $\mathfrak{A}_{k\geq 3}$, $\aleph_{k\geq 3}$

oscillate around $\sim \Lambda^2$ with $k-1$ zeros.

Just two last properties play essential role is the RS-insensitivity and in better loop convergence at low-energy region.

Better convergence of APT expansions

Instead of **e x p l o d i n g** power-in-$\bar{\alpha}_s$ series

$$d_{pt}(Q) = d_1 \bar{\alpha}_s(Q) + d_2\, \bar{\alpha}_s^2(Q) + d_3\, \bar{\alpha}_s^3(Q)\,,;$$

$$r_{pt}(s) = d_1 \bar{\alpha}_s(s) + d_2\, \bar{\alpha}_s^2(s) + r_3\, \bar{\alpha}_s^3(s)\,, \qquad r_3 = d_2 - d_1 \frac{\pi^2 \beta_0^2}{3}\,;$$

one has $r_{\mathrm{APT}}(s) = d_1 \alpha_M(s) + d_2\, \mathfrak{A}_2(s) + d_3\, \mathfrak{A}_3(s) + \ldots$

$$d_{\mathrm{APT}}(Q^2) = d_1\, \alpha_E(Q^2) + d_2\, \mathcal{A}_2(Q^2) + d_3\, \mathcal{A}_3(Q^2) + \ldots$$

Table : Relative contributions (in %) of 1– , 2– and 3–loop terms

Process		*PT*			*APT*		
GLS sum rule	t	65	24	**11**	75	21	4
Bjorken. s.r.	t	55	26	**19**	80	19	1
Incl. τ-decay	s	55	29	**16**	88	11	1

[a] NTinHEP-2006

3. Equivalence of transformations

Transition from $\alpha_s(\ell)$ to Euclidean $\alpha_E(\ell)$ e.t.c., can be treated as a transition to a new expansion parameter. E.g., $(\alpha\,, L > 0)$

$$\alpha \to \alpha_{\mathbf{M}} = \mathbf{w}_{\mathbf{1}}^{\mathbf{M}}(\alpha) = \frac{\arctan(\pi\beta_{\mathbf{0}}\,\alpha)}{\pi\,\beta_{\mathbf{0}}} \quad \sim \quad \alpha_{\mathbf{s}}(\mathbf{L}) \to \frac{\arctan\left(\pi/\mathbf{L}\right)}{\pi\,\beta_{\mathbf{0}}}\,.$$

Generally, all functions $\mathfrak{A}_k$, $\mathcal{A}_k$, $\aleph_k$ are mapped via relation

$$L \to \Phi(\alpha_s) = -\int^{\alpha_s}\frac{d\,a}{\beta(a)} = \frac{1}{\beta_0\alpha_s} + \frac{\beta_1}{\beta_0^2}\ln\left(\frac{\beta_1}{\beta_0} + \frac{1}{\alpha_s}\right) + O(\beta_2) \tag{1}$$

on sets $\{\mathbf{w}_{\mathbf{k}}^{\mathrm{APT=M,E,D}}(\alpha_{\mathbf{s}})\}$

$$\mathfrak{A}_{\mathbf{k}} \to \mathbf{w}_{\mathbf{k}}^{\mathbf{M}}(\alpha_{\mathbf{s}}), \quad \mathcal{A}_{\mathbf{k}} \to \mathbf{w}_{\mathbf{k}}^{\mathbf{E}}(\alpha_{\mathbf{s}}), \quad \aleph_{\mathbf{k}} \to \mathbf{w}_{\mathbf{k}}^{\mathbf{D}}(\alpha_{\mathbf{s}})\,.$$

1-loop functions $\mathbf{w_k^{APT}}(\alpha_s)$

There, consequent elements are connected by 1-loop recurrent relations

$$\mathbf{w}^{(1)}_{\mathbf{k+1}}(\alpha) = \frac{\alpha^{\mathbf{2}}}{\mathbf{k}} \frac{\mathbf{d\,w}^{(1)}_{\mathbf{k}}(\alpha)}{\mathbf{d}\,\alpha}\,.$$

For 1-loop case they are

$$\alpha_{\mathbf{M}} = \mathbf{w}_{\mathbf{1}}^{\mathbf{M},(1)} = \frac{\arctan(\pi\beta_{\mathbf{0}}\,\alpha)}{\pi\,\beta_{\mathbf{0}}}\,;\;\; \mathbf{w}_{\mathbf{2}}^{\mathbf{M},(1)}(\alpha) = \frac{\alpha^{\mathbf{2}}}{\mathbf{1}+\pi^{\mathbf{2}}\beta_{\mathbf{0}}^{\mathbf{2}}\,\alpha^{\mathbf{2}}}\,;$$

$$\mathbf{w}_{\mathbf{3}}^{\mathbf{M},(1)}(\alpha) = \frac{\alpha^{\mathbf{3}}}{[\mathbf{1}+(\pi\beta_{\mathbf{0}}\,\alpha)^{\mathbf{2}}]^{\mathbf{2}}}\,;\quad \mathbf{w}_{\mathbf{4}}^{\mathbf{M},(1)}(\alpha) = \frac{\alpha^{\mathbf{4}}\,(\mathbf{3}-\pi^{\mathbf{2}}\,\beta_{\mathbf{0}}^{\mathbf{2}}\,\alpha^{\mathbf{2}})}{\mathbf{3}\,[\mathbf{1}+(\pi\beta_{\mathbf{0}}\,\alpha)^{\mathbf{2}}]^{\mathbf{3}}}\,;$$

$$\mathbf{w}_{\mathbf{1}}^{\mathbf{E},(1)}(\alpha) = \alpha + \frac{\beta_{\mathbf{0}}^{-\mathbf{1}}}{\mathbf{1}-\mathbf{e}^{\mathbf{1}/(\beta_{\mathbf{0}}\alpha)}}\,;\quad \mathbf{w}_{\mathbf{2}}^{\mathbf{E},(1)}(\alpha) = \alpha^{\mathbf{2}} - \frac{\beta_{\mathbf{0}}^{-\mathbf{2}}\,\mathbf{e}^{\mathbf{1}/(\beta_{\mathbf{0}}\alpha)}}{\left[\mathbf{1}-\mathbf{e}^{\mathbf{1}/(\beta_{\mathbf{0}}\alpha)}\right]^{\mathbf{2}}}\,;\;\ldots$$

1-loop functions $\mathbf{w}_{\mathbf{k}}^{\cdots}(\alpha_{\mathbf{s}})$ at $\alpha_s \to \infty$

At $\alpha_s \to \infty$ finite limits:

$$\alpha_{\mathbf{M}}(\infty) = \alpha_{\mathbf{E}}(\infty) = \frac{\mathbf{1}}{\mathbf{2}\,\beta_{\mathbf{0}}} \sim \mathbf{0.7}\,;$$

$$\mathbf{w}_{\mathbf{2}}^{\mathbf{M},(\mathbf{1})}(\infty) = \frac{\mathbf{1}}{\pi^{\mathbf{2}}\beta_{\mathbf{0}}^{\mathbf{2}}} \sim \mathbf{0.20}\,; \qquad \mathbf{w}_{\mathbf{2}}^{\mathbf{E},(\mathbf{1})}(\infty) = \frac{\mathbf{1}}{\mathbf{12}\beta_{\mathbf{0}}^{\mathbf{2}}} \sim \mathbf{0.16}\,;$$

$$\mathbf{w}_{\mathbf{3}}^{\mathbf{M},(\mathbf{1})}(\infty) = \mathbf{0}\,; \qquad \mathbf{w}_{\mathbf{4}}^{\mathbf{M},(\mathbf{1})}(\alpha) = -\frac{\mathbf{1}}{(\pi\,\beta_{\mathbf{0}})^{\mathbf{4}}} \sim -\mathbf{0.04}\,.$$

Here, limit $\alpha_s \to \infty$, by (1), corresponds to $L \to 0$. Curious enough, region of $L < 0$ corresponds to negative α_s values.

Properties of $\{\mathbf{w}_{\mathbf{k}}^{\cdots}(\alpha_{\mathbf{s}})\}$

Comments on new expansion functions:

- Like in APT, $\{\mathbf{w}_{\mathbf{k}}^{\cdots}(\alpha_{\mathbf{s}})\}$ - $(-\infty < \alpha_s < +\infty)$ - form non-power oscillating sets with k zeroes.
- Natural scales are $\alpha_M^* = \frac{1}{\pi\beta_0} \sim \mathbf{0.5}$, $\alpha_E^* = \frac{1}{\beta_0} \sim 1.4$.
- Some, like $\mathbf{w}_{\mathbf{k}}^{\mathbf{E}}$, obey singularity $e^{(1/\beta_0\,\alpha_s)}$.
- They are not sensitive to “their origin". In Fig.4, curves $\mathbf{w}_{\mathbf{1,2}}^{\mathbf{E}}$ are close to $\mathbf{w}_{\mathbf{1,2}}^{\mathbf{M}}$ (comparing with Caprini-Fischer [conformal-improved Borel image (2000)] curves $w_{1,2}^{CF}$).

[a] NTinHEP-2006

Comparing $\mathbf{w}^{\mathbf{E}}_{\mathbf{1,2}}$ with $\mathbf{w}^{\mathbf{M}}_{\mathbf{1,2}}$ and $w^{cf}_{1,2}$

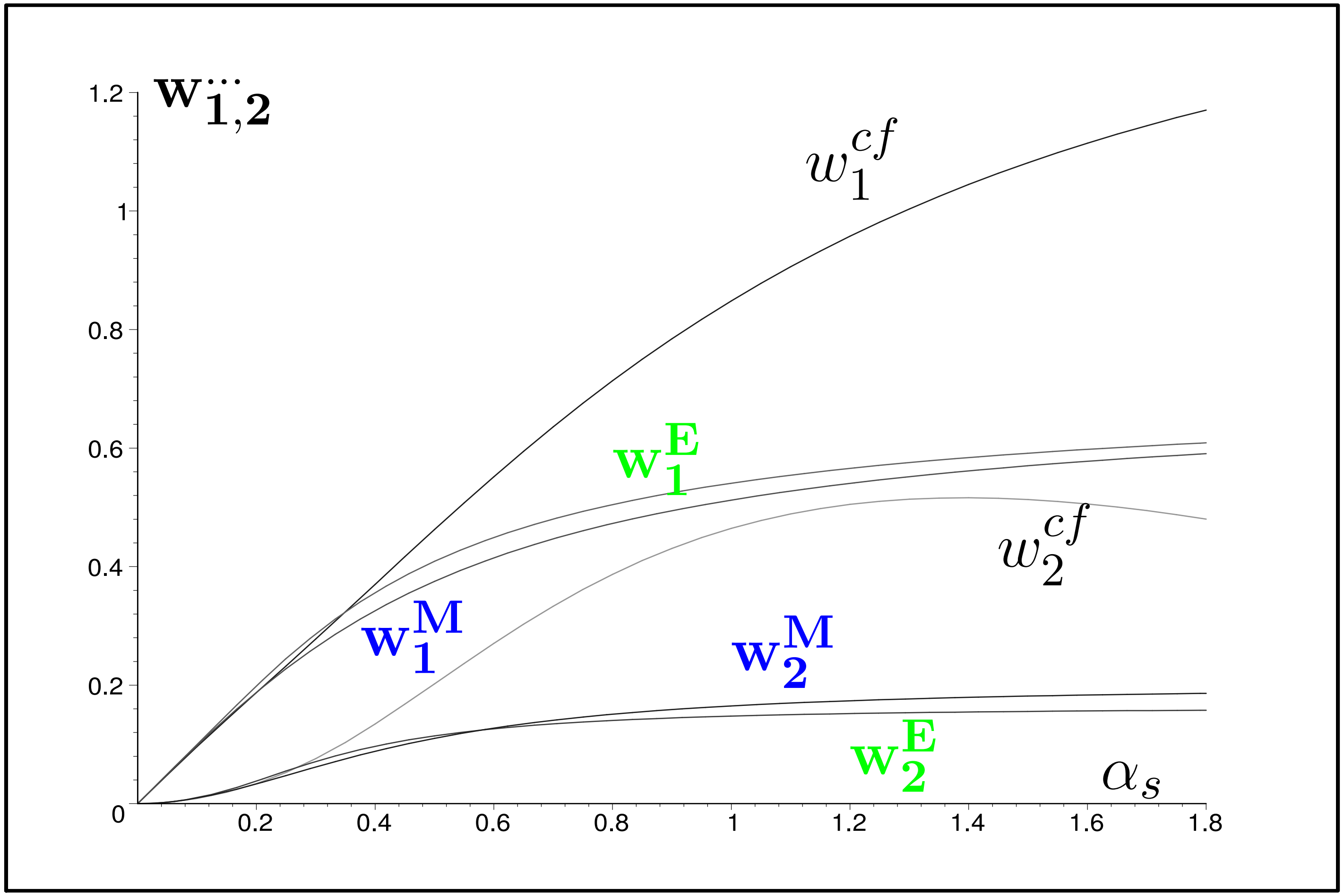

Figure 4: *Comparison of first two APT-inspired $\mathbf{w}^{\mathbf{E,M}}_{\mathbf{1,2}}(\alpha_{\mathbf{s}})$ functions with Caprini-Fischer $w^{cf}_{1,2}$ ones.*

Properties of $\{\mathbf{w}_{\mathbf{k}}^{\mathbf{APT}}(\alpha_{\mathbf{s}})\}$

1. APT-inspired $\mathbf{w}_{\mathbf{1,2}}^{\mathbf{APT}}(\alpha_{\mathbf{s}})$ deviate from powers α_s, α_s^2 at $\alpha_s \sim 0.3 - 0.4$, *i.e.* , in few GeV region.

2. Numerical difference between various APT functions is small up to $\alpha_s \sim 0.8$.

3. Quick saturation of first $\mathbf{w}_1(\alpha_{\mathbf{s}})$ at 0.4 and 2nds $\mathbf{w}_2(\alpha_{\mathbf{s}})$ at $\sim$ 0.15 values.

Strong coupling means $\alpha_{\mathbf{eff}} \lesssim \mathbf{0.4}$.

[a] NTinHEP-2006

Possible use of $\{\mathbf{w}_{\mathbf{k}}^{\mathbf{APT}}(\alpha_{\mathbf{s}})\}$

4. Usual RG-improved PT is limited by "1-argument" objects [total cross-sections]. For "2-argument" ones [diffraction amplitudes, structure functions], one uses additional tricks, *e.g.* , consider 1-arg. moments.

5. New RG-inspired expansions over $(\alpha_{\mathrm{s}})^{\mathrm{k}} \to \mathbf{w}_{\mathbf{k}}^{\mathrm{APT}}(\alpha_{\mathrm{s}})$ provide another bypass solution to this issue.

(Sorry for **Terazawa et others 1965-??**)

[a] NTinHEP-2006